

Colored HOMFLY polynomials from Chern-Simons theory

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ABSTRACT: We elaborate the Chern-Simons field theoretic method to obtain colored HOMFLY invariants of knots and links. Using multiplicity-free quantum $6j$ -symbols for $U_q(\mathfrak{sl}_N)$, we present explicit evaluations of the HOMFLY invariants colored by symmetric representations for a variety of knots, two-component links and three-component links.

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1 Introduction

For the last few decades, we have seen tremendous developments on knot theory, a subject where diverse areas in mathematics and physics interact in beautiful ways. The interplay between mathematics and physics involving knot theory was triggered by the seminal paper of Witten [1] which shows that Chern-Simons theory provides a natural framework to study link invariants. In particular, the expectation value of Wilson loop along a link \mathcal{L} in S^3 gives a topological invariant of the link depending on the representation of the gauge group. For a representation R of $SU(2)$ gauge group, an invariant corresponds to a colored Jones polynomial $J_R(\mathcal{L}; q)$. Besides, one can relate an $SU(N)$ invariant with representation R to a colored HOMFLY invariant $P_R(\mathcal{L}; a, q)$. While the systematic procedure to compute $SU(N)$ invariants in S^3 is investigated in [2–4], it is very difficult to carry out explicit computations in general. Even in mathematics, although the definition [5, 6] of colored HOMFLY polynomials was provided, explicit calculations for non-trivial knots and links are far from under control.

Nevertheless, in recent years, there have been spectacular developments on computations of colored HOMFLY polynomials. For torus knots and links, the HOMFLY invariants colored by arbitrary representations can be, in principle, computed by using the generalizations [6–8] of the Rosso-Jones formulae [9]. In addition, Kawagoe has lately formulated a mathematically rigorous procedure based on the linear skein theory to calculate HOMFLY invariants colored by symmetric representations for some non-torus knots and links [10]. Furthermore, the explicit closed formulae of the colored HOMFLY polynomials $P_{[n]}(\mathcal{K}; a, q)$ with symmetric representations $(\overleftarrow{\square} \cdot \overrightarrow{\square})$ were provided for the $\mathcal{K} = (2, 2p+1)$ -torus knots [11] and the twist knots [10, 12, 13].

In this paper, we shall demonstrate the computations of the HOMFLY polynomials colored by symmetric representations in the framework of Chern-Simons theory. Exploiting the connection between Chern-Simons theory and the two-dimensional Wess-Zumino-Novikov-Witten (WZNW) model, the prescription to evaluate expectation values of Wilson loops was formulated entirely in terms of the fusion and braid operations on conformal blocks of the WZNW model [2–4]. Therefore, the procedure inevitably involves the $SU(N)$ quantum Racah coefficients (the quantum $6j$ -symbols for $U_q(\mathfrak{sl}_N)$), which makes explicit computations hard. The first step along this direction has been made in [14]: using the properties the $SU(N)$ quantum Racah coefficients should obey, the explicit expressions involving first few symmetric representations are determined. This result as well as the closed formulae of the twist knots motivated us to explore a closed form expression for the $SU(N)$ quantum Racah coefficients. We succeeded in writing the expression for multiplicity-free representations [15] which enables us to compute the colored HOMFLY polynomials carrying symmetric representations. To consider knots and links more complicated than the ones treated in [14], we make use of cut-and-gluing of S^3 developed in [2, 3].

With this method, the expressions of the twist knots, the Whitehead links, the twist links and the Borromean rings [10, 12, 16] have been reproduced up to 4 boxes. Even apart from these classes of knots and links, the validity of our procedure is checked from the complete agreement with the results obtained in [17, 18]. Furthermore, the explicit

evaluations of multi-colored link invariants shed a new light on the general properties of colored HOMFLY invariants of links and provide meaningful implications on homological invariants of links.

The plan of the paper is as follows. In §2, we briefly review $U(N)$ Chern-Simons theory. In particular, we present the list of building blocks and the corresponding states which are necessary for calculations of colored HOMFLY polynomials. In §3, we compute the colored HOMFLY polynomials of seven-crossing knots and ten-crossing thick knots. In §4, multi-colored HOMFLY invariants for two-component and three-component links are expressed. We summarize and present few open problems in §5. For convenience, we explicitly show $SU(N)$ quantum the Racah coefficients for some representations in Appendix A. Finally, we should mention that a Mathematica file with colored HOMFLY invariants whose expressions are too lengthy for the main text is linked on the arXiv page as an ancillary file.

2 Chern-Simons theory invariants of knots and links

We shall briefly discuss $U(N)$ Chern-Simons theory necessary for computing invariants of framed knots and framed links. The action for $U(N) \simeq U(1) \times SU(N)$ Chern-Simons theory is given by

$$S = \frac{k_1}{4\pi} \int_{S^3} B \wedge dB + \frac{k}{4\pi} \int_{S^3} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (2.1)$$

where B is the $U(1)$ gauge connection and A is the $SU(N)$ matrix valued gauge connection with Chern-Simons coupling (also referred as Chern-Simons level) k_1 and k respectively. The Wilson loop observable for an arbitrary framed link \mathcal{L} made up of s -components $\{\mathcal{K}_\beta\}$, with framing number f_β , is the trace of the holonomies of the components:

$$W_{(R_1, n_1), (R_2, n_2), \dots, (R_s, n_s)}[\mathcal{L}] = \prod_{\beta=1}^s \text{Tr}_{R_\beta} U^A[\mathcal{K}_\beta] \text{Tr}_{n_\beta} U^B[\mathcal{K}_\beta], \quad (2.2)$$

where the holonomy of the gauge field A around a component knot \mathcal{K}_β , carrying a representation R_β of an s -component link is denoted by $U^A[\mathcal{K}_\beta] = P[\exp \oint_{\mathcal{K}_\beta} A]$ and n_β is the $U(1)$ charge carried by the component knot \mathcal{K}_β . Note that the framing number f_β for the component knot \mathcal{K}_β is the difference between the total number of left-handed crossings and that of right-handed crossings. The expectation values of these Wilson loop operators are the framed link invariants:

$$V_{R_1, \dots, R_s}^{\{SU(N)\}}[\mathcal{L}] V_{n_1, \dots, n_s}^{\{U(1)\}}[\mathcal{L}] = \langle W_{(R_1, n_1), \dots, (R_s, n_s)}[\mathcal{L}] \rangle = \frac{\int [\mathcal{D}B][\mathcal{D}A] e^{iS} W_{(R_1, n_1), \dots, (R_s, n_s)}[\mathcal{L}]}{\int [\mathcal{D}B][\mathcal{D}A] e^{iS}}. \quad (2.3)$$

The $SU(N)$ invariants will be rational functions in the variable $q = \exp\left(\frac{2\pi i}{k+N}\right)$ [19, 20] with the following choice for $U(1)$ charge and coupling k_1 ,

$$n_\beta = \frac{\ell^{(\beta)}}{\sqrt{N}}; \quad k_1 = k + N, \quad (2.4)$$

where $\ell^{(\beta)}$ is the total number of boxes in the Young Tableau representation R_β . The $U(1)$ invariant involves only linking numbers $\{\text{Lk}_{\alpha\beta}\}$ between the component knots and the framing numbers $\{f_\beta\}$ of each component knot. That is,

$$V_{\frac{\ell^{(1)}}{\sqrt{N}}, \dots, \frac{\ell^{(s)}}{\sqrt{N}}}^{\{U(1)\}}[\mathcal{L}] = (-1)^{\sum_\beta \ell^{(\beta)} f_\beta} \exp\left(\frac{i\pi}{k+N} \sum_{\beta=1}^s \frac{(\ell^{(\beta)})^2 f_\beta}{N}\right) \exp\left(\frac{i\pi}{k+N} \sum_{\alpha \neq \beta} \frac{\ell^{(\alpha)} \ell^{(\beta)} \text{Lk}_{\alpha\beta}}{N}\right). \quad (2.5)$$

The $SU(N)$ invariants can be obtained non-perturbatively using these ingredients:

1. The connection between $SU(N)$ Chern-Simons integral on a three-ball to the $SU(N)_k$ WZNW model on the boundary [1].
2. Any knot or a link can be drawn as a closure (Alexander theorem) or platting of braids [21].
3. Redraw equivalent diagrams of the braids inside the three-ball such that the knot in S^3 can be seen as gluing of many three-balls each with several four-punctured S^2 boundaries [2–4]. See Figure 3.

The Chern-Simons functional integral over a three-ball, with four-punctured S^2 boundary, is given by a state in the Hilbert space spanned by four-point conformal blocks. There are two different four-point conformal block bases as shown in Figure 1 where the internal

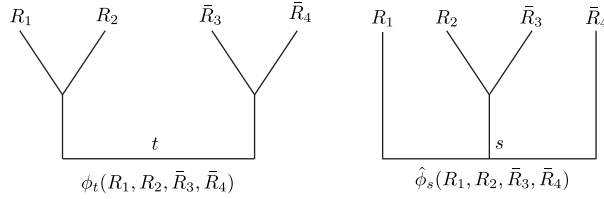


Figure 1. Four-point conformal block bases states

representations satisfy the fusion rules $t \in (R_1 \otimes R_2) \cap (R_3 \otimes R_4)$ and $s \in (R_2 \otimes \bar{R}_3) \cap (\bar{R}_1 \otimes R_4)$. The conformal block $|\phi_t(R_1, R_2, \bar{R}_3, \bar{R}_4)\rangle$ is suitable for the braiding operators $b_1^{(\pm)}$ and $b_3^{(\pm)}$ where b_i denotes right-handed half-twist or braiding between the i^{th} and the $(i+1)^{\text{th}}$ strand. Here the superscripts $(+)$ and $(-)$ denote the braidings on two strands in parallel orientations and in anti-parallel orientations respectively. Similarly, the braiding in the middle two strands involving the operator $b_2^{(\pm)}$ requires the conformal block $|\hat{\phi}_s(R_1, R_2, \bar{R}_3, \bar{R}_4)\rangle$. That is,

$$\begin{aligned} b_1^{(\pm)} |\phi_t(R_1, R_2, \bar{R}_3, \bar{R}_4)\rangle &= \lambda_t^{(\pm)}(R_1, R_2) |\phi_t(R_2, R_1, \bar{R}_3, \bar{R}_4)\rangle, \\ b_2^{(\pm)} |\hat{\phi}_s(R_1, R_2, \bar{R}_3, \bar{R}_4)\rangle &= \lambda_s^{(\pm)}(R_2, \bar{R}_3) |\hat{\phi}_s(R_1, \bar{R}_3, R_2, \bar{R}_4)\rangle, \\ b_3^{(\pm)} |\phi_t(R_1, R_2, \bar{R}_3, \bar{R}_4)\rangle &= \lambda_t^{(\pm)}(\bar{R}_3, \bar{R}_4) |\phi_t(R_1, R_2, \bar{R}_4, \bar{R}_3)\rangle, \end{aligned} \quad (2.6)$$

where the braiding eigenvalues $\lambda_t^{(\pm)}(R_1, R_2)$ in the vertical framing are

$$\lambda_t^{(\pm)}(R_1, R_2) = \epsilon_{t; R_1, R_2}^{(\pm)} \left(q^{\frac{C_{R_1} + C_{R_2} - C_{R_t}}{2}} \right)^{\pm 1}, \quad (2.7)$$

where $\epsilon_{t;R_1,R_2}^{(\pm)} = \pm 1$ (See (3.9) in [14]). Here the quadratic Casimir for the representation R is denoted by C_R :

$$C_R = \kappa_R - \frac{\ell^2}{2N}, \quad \kappa_R = \frac{1}{2}[N\ell + \ell + \sum_i (\ell_i^2 - 2i\ell_i)], \quad (2.8)$$

where ℓ_i is the number of boxes in the i^{th} row of the Young Tableau corresponding to the representation R and ℓ is the total number of boxes. The two bases in Figure 1 are related by a fusion matrix a_{st} as follows:

$$|\phi_t(R_1, R_2, \bar{R}_3, \bar{R}_4)\rangle = a_{ts} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_3 & \bar{R}_4 \end{bmatrix} |\hat{\phi}_s(R_1, R_2, \bar{R}_3, \bar{R}_4)\rangle. \quad (2.9)$$

The fusion matrix is determined by the $SU(N)$ quantum Racah coefficients. We have obtained these coefficients for few representations [14] and recently for symmetric representations [15].

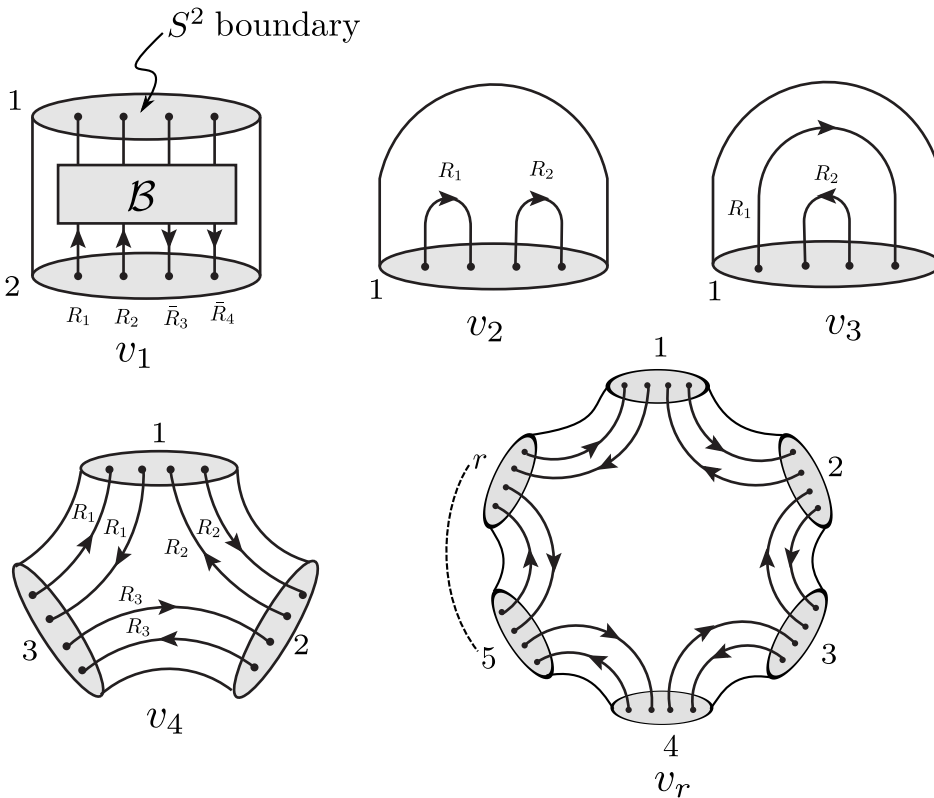


Figure 2. Fundamental building blocks

In order to write the explicit polynomial form of $SU(N)$ invariants for many knots, we will require the states corresponding to Chern-Simons functional integral over three-balls with several four-punctured S^2 boundaries. Therefore, we will present these states which will serve as the necessary building blocks for the knots and links in §3 and §4. Inside a three-ball with two S^2 boundaries, we have a four-strand braid with braid word \mathcal{B} as shown

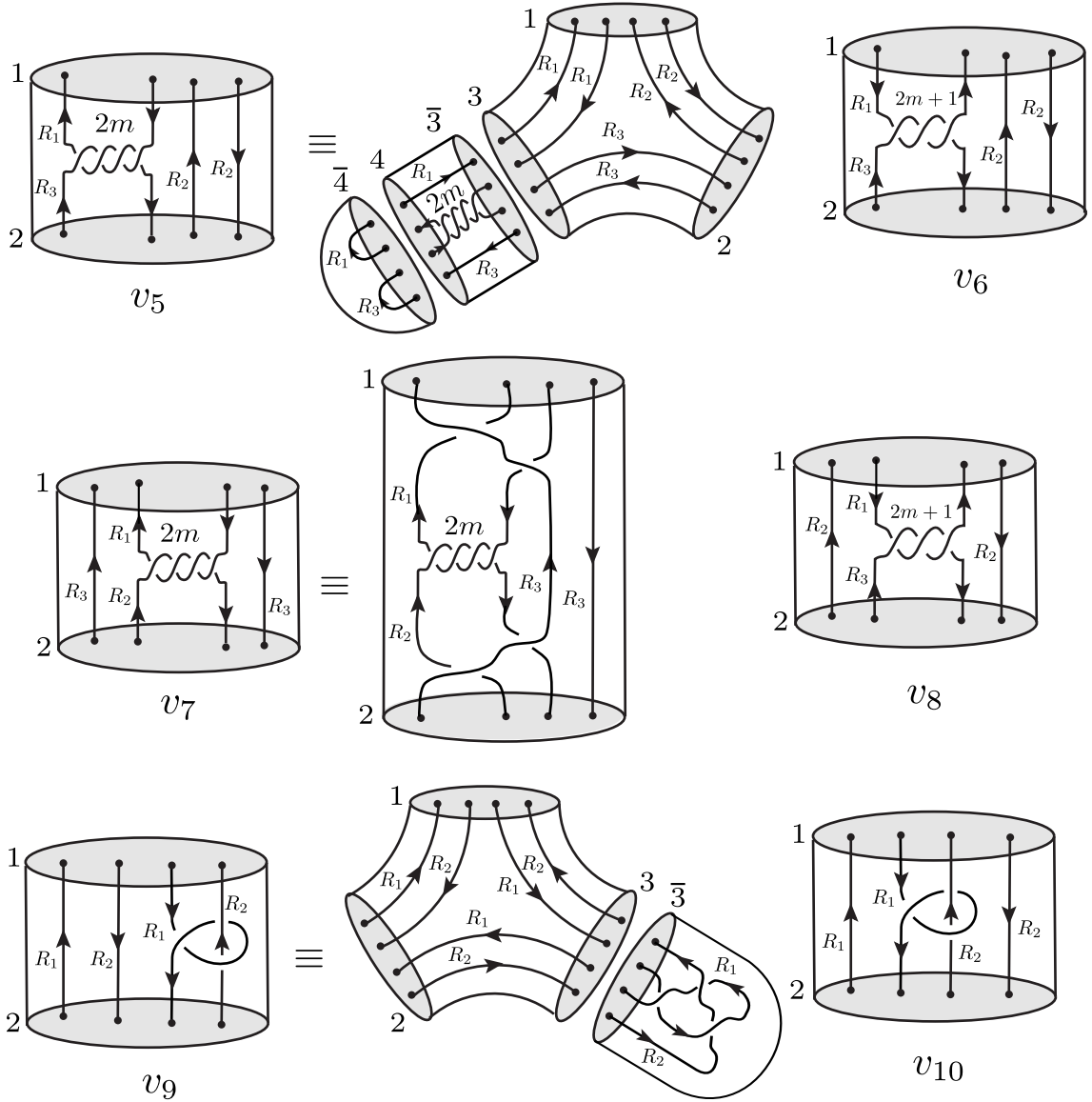


Figure 3. Composite building blocks

by v_1 in Figure 2. We shall call the state as v_1 whose form in terms of four-point conformal blocks of the WZNW model is

$$v_1 = \sum_{l \in (R_1 \otimes R_2) \cap (R_3 \otimes R_4)} \{ \mathcal{B} | \phi_l(R_1, R_2, \bar{R}_3, \bar{R}_4) \rangle \}^{(1)} | \phi_l(R_1, R_2, \bar{R}_3, \bar{R}_4) \rangle^{(2)}, \quad (2.10)$$

where the superscripts outside the four-point conformal blocks denotes the boundaries (1) and (2) as indicated in the Figure 2. For the simplest three-ball with one S^2 boundary, the states v_2 and v_3 will be

$$v_2 = | \phi_0(R_1, \bar{R}_1, R_2, R_2) \rangle^{(1)}, \quad v_3 = | \hat{\phi}_0(R_1, \bar{R}_2, R_2, \bar{R}_1) \rangle^{(1)}, \quad (2.11)$$

where the subscript 0 in ϕ_0 represents the singlet representation. This procedure can be generalized to three-ball with more than one S^2 boundary. For definiteness, we first write the state v_4 for three S^2 boundaries and generalize to state v_r for r S^2 boundaries:

$$v_4 = \sum_{l \in (R_1 \otimes \bar{R}_1) \cap (R_2 \otimes \bar{R}_2) \cap (R_3 \otimes \bar{R}_3)} \frac{1}{\epsilon_l \sqrt{\dim_q l}} |\phi_l(\bar{R}_1, R_1, \bar{R}_2, R_2)\rangle^{(1)} \\ |\phi_l(\bar{R}_2, R_2, \bar{R}_3, R_3)\rangle^{(2)} |\phi_l(\bar{R}_3, R_3, \bar{R}_1, R_1)\rangle^{(3)}, \\ v_r = \sum_l \frac{1}{(\epsilon_l \sqrt{\dim_q l})^{\frac{r-2}{2}}} |\phi_l(\bar{R}_1, R_1, \bar{R}_2, R_2)\rangle^{(1)} \dots |\phi_l(\bar{R}_r, R_r, \bar{R}_1, R_1)\rangle^{(r)}. \quad (2.12)$$

Here $\epsilon_l \equiv \epsilon_l^{R_1, \bar{R}_1} = \pm 1$ (See (3.1) in [14]). Using these fundamental building blocks, we can obtain states for three-ball with two S^2 boundaries in Figure 3 which we call composite building blocks. For example, the state v_5 for the two S^2 boundaries can be viewed as gluing of appropriate oppositely oriented boundaries of v_1 , v_2 and v_4 as shown. In the equivalent diagram for v_5 , we have indicated $\bar{3}$ on an S^2 boundary which denotes that it is oppositely oriented to the S^2 boundary numbered 3. Gluing along two oppositely oriented S^2 boundaries amounts to taking inner product of the states corresponding to the boundaries. For example, gluing along the S^2 boundaries 3 and $\bar{3}$ results in

$$\langle \bar{3} | \phi_l(R_3, \bar{R}_3, R_1, \bar{R}_1) | \phi_x(\bar{R}_3, R_3, \bar{R}_1, R_1) \rangle^{(3)} = \delta_{lx}. \quad (2.13)$$

Writing the states v_1 , v_2 and v_4 and taking appropriate inner product, we obtain the state v_5 as

$$v_5 = \sum_{l,r} \frac{1}{\epsilon_l \sqrt{\dim_q l}} \epsilon_r^{\bar{R}_1, R_3} \sqrt{\dim_q r} (\lambda_r^{(-)}(\bar{R}_1, R_3))^{2m} a_{lr} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_3 & \bar{R}_3 \end{bmatrix} \\ \times |\phi_l(\bar{R}_1, R_1, \bar{R}_2, R_2)\rangle^{(1)} |\phi_l(R_3, \bar{R}_3, R_2, \bar{R}_2)\rangle^{(2)}. \quad (2.14)$$

The state v_6 is similar to the state v_5 , but it involves odd number of braidings:

$$v_6 = \sum_{l,r} \frac{1}{\epsilon_l^{\bar{R}_1, R_3} \sqrt{\dim_q l}} \epsilon_r^{R_1, R_3} \sqrt{\dim_q r} (\lambda_r^{(+)}(R_1, R_3))^{-(2m+1)} a_{rl} \begin{bmatrix} R_1 & R_3 \\ \bar{R}_1 & \bar{R}_3 \end{bmatrix} \\ \times |\phi_l(R_1, \bar{R}_3, \bar{R}_2, R_2)\rangle^{(1)} |\phi_l(R_3, \bar{R}_1, R_2, \bar{R}_2)\rangle^{(2)}. \quad (2.15)$$

The state v_7 can be obtained by gluing v_1 , v_5 and again a v_1 with appropriate braid words \mathcal{B} in both v_1 's as shown in the equivalent diagram:

$$v_7 = \sum_{l,r,x,y} \frac{1}{\epsilon_l \sqrt{\dim_q l}} \epsilon_r^{\bar{R}_1, R_3} \sqrt{\dim_q r} (\lambda_r^{(-)}(\bar{R}_1, R_3))^{2m} a_{lr} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_3 & \bar{R}_3 \end{bmatrix} \\ \times a_{lx} \begin{bmatrix} R_1 & \bar{R}_1 \\ \bar{R}_3 & R_3 \end{bmatrix} a_{ly} \begin{bmatrix} \bar{R}_2 & R_2 \\ R_3 & \bar{R}_3 \end{bmatrix} |\phi_x(\bar{R}_3, \bar{R}_1, R_1, R_3)\rangle^{(1)} |\phi_y(R_3, R_2, \bar{R}_2, \bar{R}_3)\rangle^{(2)}. \quad (2.16)$$

The state v_8 is almost similar to the state v_7 except for the odd number of braidings instead of even number of braidings:

$$v_8 = \sum_{l,r,x,y} \frac{1}{\epsilon_l^{\bar{R}_1, R_3} \sqrt{\dim_q l}} \epsilon_r^{R_1, R_3} \sqrt{\dim_q r} (\lambda_r^{(+)}(R_1, R_3))^{-(2m+1)} a_{rl} \begin{bmatrix} R_1 & R_3 \\ \bar{R}_1 & \bar{R}_3 \end{bmatrix}$$

$$\times a_{x\bar{l}} \begin{bmatrix} \bar{R}_2 & R_1 \\ \bar{R}_3 & R_2 \end{bmatrix} a_{yl} \begin{bmatrix} R_2 & R_3 \\ \bar{R}_1 & \bar{R}_2 \end{bmatrix} |\phi_x(\bar{R}_2, R_1, \bar{R}_3, R_2)\rangle^{(1)} |\phi_y(R_2, R_3, \bar{R}_1, \bar{R}_2)\rangle^{(2)}. \quad (2.17)$$

The equivalent diagram for v_9 in Figure 3 determines the state as

$$\begin{aligned} v_9 = & \sum_{l,x,y,z} \frac{1}{\epsilon_l^{\bar{R}_1, R_2} \sqrt{\dim_q l}} \epsilon_z^{\bar{R}_1, R_2} \sqrt{\dim_q z} a_{xl} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} a_{yx} \begin{bmatrix} R_2 & R_1 \\ \bar{R}_1 & \bar{R}_2 \end{bmatrix} \\ & \times a_{zy} \begin{bmatrix} \bar{R}_1 & R_2 \\ R_1 & \bar{R}_2 \end{bmatrix} \lambda_x^{(-)}(R_1, \bar{R}_1) \lambda_y^{(+)}(R_1, R_2) \lambda_z^{(-)}(\bar{R}_1, R_2) |\phi_l(\bar{R}_1, R_2, R_1, \bar{R}_2)\rangle^{(1)} \\ & \times |\phi_{\bar{l}}(R_1, \bar{R}_2, \bar{R}_1, R_2)\rangle^{(2)}. \end{aligned} \quad (2.18)$$

To get the state v_{10} , we could glue the state v_1 to the state v_9 with braid words $\mathcal{B} = b_2^{(+)} \{b_3^{(-)}\}^{-1}$:

$$\begin{aligned} v_{10} = & \sum_{l,x,y,z} \frac{1}{\epsilon_l^{\bar{R}_1, R_2} \sqrt{\dim_q l}} \epsilon_z^{\bar{R}_1, R_2} \sqrt{\dim_q z} a_{xl} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} a_{yx} \begin{bmatrix} R_2 & R_1 \\ \bar{R}_1 & \bar{R}_2 \end{bmatrix} \\ & \times a_{zy} \begin{bmatrix} \bar{R}_1 & R_2 \\ R_1 & \bar{R}_2 \end{bmatrix} a_{sl} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} a_{tl} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} \lambda_x^{(-)}(R_1, \bar{R}_1) \lambda_y^{(+)}(R_1, R_2) \\ & \times \lambda_z^{(-)}(\bar{R}_1, R_2) |\phi_s(\bar{R}_1, R_1, \bar{R}_2, R_2)\rangle^{(1)} |\phi_t(R_1, \bar{R}_1, R_2, \bar{R}_2)\rangle^{(2)}. \end{aligned} \quad (2.19)$$

Our main aim is to redraw many knots and links in S^3 using these building blocks so that the invariant involves only the multiplicity-free Racah coefficients. For instance, see Figure 6 and Figure 7 in [22] where equivalent diagrams of the knots **9₄₂** and **10₇₁** are drawn.

With these building blocks as tools, we will demonstrate the evaluation of the Chern-Simons invariant for the knot **10₁₅₂**. This knot can be viewed as gluing of five three-balls as shown in Figure 4. Using the states for the fundamental and composite building blocks, we can directly write the different states corresponding to the three-balls $\{p_i\}$'s as follows:

$$p_1 = \sum_{s_1, u} \epsilon_u^{R, \bar{R}} \sqrt{\dim_q u} a_{s_1 u} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} (\lambda_u^{(-)}(R, \bar{R}))^2 |\phi_{s_1}(\bar{R}, R, \bar{R}, R)\rangle^{(1)} \quad (2.20)$$

$$p_2 = \sum_l \frac{1}{\epsilon_l^{R, \bar{R}} \sqrt{\dim_q l}} |\phi_l(R, \bar{R}, R, \bar{R})\rangle^{(1)} |\phi_l(R, \bar{R}, R, \bar{R})\rangle^{(2)} |\phi_l(\bar{R}, R, \bar{R}, R)\rangle^{(3)} \quad (2.21)$$

$$p_3 = \sum_{s_2, v} \epsilon_v^{R, R} \sqrt{\dim_q v} a_{s_2 v} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} \lambda_{s_2}^{(-)}(R, \bar{R}) (\lambda_v^{(+)}(R, R))^3 |\phi_{s_2}(\bar{R}, R, \bar{R}, R)\rangle^{(2)} \quad (2.22)$$

$$\begin{aligned} p_4 = & \sum_{l_1, r, x, y} \frac{1}{\epsilon_{l_1}^{R, \bar{R}} \sqrt{\dim_q l_1}} \epsilon_r^{R, R} \sqrt{\dim_q r} (\lambda_r^{(+)}(R, R))^3 a_{r l_1} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} \\ & \times a_{x l_1} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{y l_1} \begin{bmatrix} \bar{R} & \bar{R} \\ R & R \end{bmatrix} |\phi_x(R, \bar{R}, R, \bar{R})\rangle^{(3)} |\phi_y(\bar{R}, \bar{R}, R, R)\rangle^{(4)} \end{aligned} \quad (2.23)$$

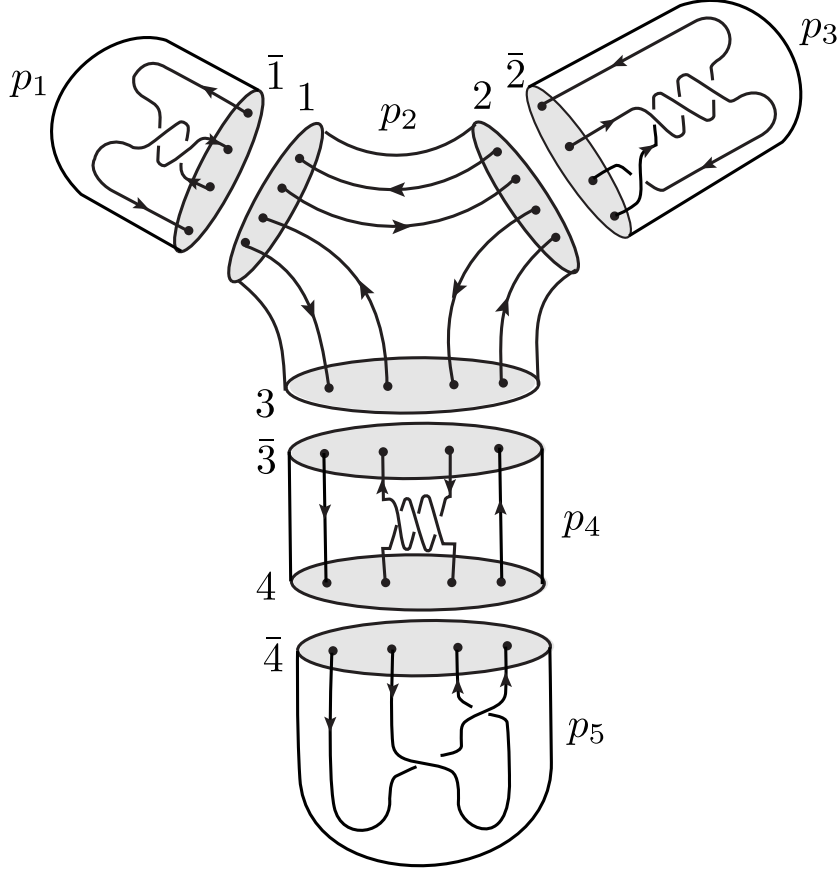


Figure 4. The knot 10_{152} in gluing of three-balls.

$$p_5 = \sum_{s_3, z} \epsilon_z^{R, \bar{R}} \sqrt{\dim_q z} a_{s_3 z} \left[\begin{array}{c} R \ R \\ \bar{R} \ \bar{R} \end{array} \right] \lambda_{s_3}^{(+)}(R, R) \lambda_z^{(-)}(R, \bar{R}) |\phi_{s_3}(R, R, \bar{R}, \bar{R})\rangle^{(4)} \quad (2.24)$$

Then gluing all the three-balls together amounts to taking appropriate inner product of the above five states on oppositely oriented S^2 boundaries resulting in the $SU(N)$ invariant

$$\begin{aligned} V_R^{\{SU(N)\}}[10_{152}] = & \sum_{l, l_1, r, u, v, x, y, z} \frac{1}{\epsilon_l^{R, \bar{R}} \sqrt{\dim_q l} \epsilon_{l_1}^{R, \bar{R}} \sqrt{\dim_q l_1}} \epsilon_z^{R, \bar{R}} \sqrt{\dim_q z} \\ & \times \epsilon_u^{R, \bar{R}} \sqrt{\dim_q u} \epsilon_r^{R, R} \sqrt{\dim_q r} \epsilon_v^{R, R} \sqrt{\dim_q v} \\ & \times a_{rl_1} \left[\begin{array}{c} R \ R \\ \bar{R} \ \bar{R} \end{array} \right] a_{ll_1} \left[\begin{array}{c} R \ \bar{R} \\ R \ \bar{R} \end{array} \right] a_{yl_1} \left[\begin{array}{c} R \ R \\ \bar{R} \ \bar{R} \end{array} \right] a_{yz} \left[\begin{array}{c} R \ R \\ \bar{R} \ \bar{R} \end{array} \right] a_{lu} \left[\begin{array}{c} R \ \bar{R} \\ R \ \bar{R} \end{array} \right] \\ & \times a_{vl} \left[\begin{array}{c} R \ R \\ \bar{R} \ \bar{R} \end{array} \right] (\lambda_r^{(+)}(R, R))^3 \lambda_y^{(+)}(R, R) \lambda_z^{(-)}(R, \bar{R}) \\ & \times (\lambda_u^{(-)}(R, \bar{R}))^2 \lambda_l^{(-)}(R, \bar{R}) (\lambda_v^{(+)}(R, R))^3. \end{aligned} \quad (2.25)$$

The framing number for the knot $\mathbf{10}_{152}$ as drawn in Figure 4 is $f = -11$ giving $U(1)$ invariant (2.5):

$$V_R^{\{U(1)\}}[\mathbf{10}_{152}] = q^{\frac{-11\ell^2}{2N}} . \quad (2.26)$$

We introduce additional twists with framing number \tilde{f} in the knot so that the framing number becomes zero ($f + \tilde{f} = 0$). Hence the $U(1)$ invariant is one for zero framed knots. This additional twist leads to multiplying $SU(N)$ invariant by a factor $q^{\tilde{f}C_R}$ where $\tilde{f} = 11$ for the knot $\mathbf{10}_{152}$ giving the unreduced HOMFLY polynomial

$$\bar{P}_R(\mathbf{10}_{152}; a = q^N, q) = q^{11C_R} V_R^{\{U(1)\}}[\mathbf{10}_{152}] V_R^{\{SU(N)\}}[\mathbf{10}_{152}] . \quad (2.27)$$

The reduced HOMFLY polynomial is expressed

$$P_R(\mathbf{10}_{152}; a, q) = \bar{P}_R(\mathbf{10}_{152}; a, q) / \bar{P}_R(\bigcirc; a, q) , \quad (2.28)$$

where the unknot factor carrying the rank- n symmetric representation is

$$\bar{P}_{[n]}(\bigcirc; a, q) = \frac{q^{n/2}(a; q)_n}{a^{n/2}(q; q)_n} . \quad (2.29)$$

We denote the q -Pochhammer symbols by $(z; q)_k = \prod_{j=0}^{k-1} (1 - zq^j)$. The factor $q^{\tilde{f}C_R}$ can be incorporated as framing correction in the vertical framing braiding eigenvalues:

$$\hat{\lambda}_s^{(+)}(R, R) = q^{C_R} \lambda_s^{(+)}(R, R) , \quad \hat{\lambda}_s^{(-)}(R, \bar{R}) = q^{C_R} \lambda_s^{(-)}(R, \bar{R}) , \quad (2.30)$$

where $\hat{\lambda}$ denotes standard framing eigenvalues which will be used in the explicit computation of colored HOMFLY polynomials of knots.

3 Colored HOMFLY polynomials for knots

In this section, we shall demonstrate computations of the colored HOMFLY polynomials of knots. The closed form expressions of the colored HOMFLY polynomials with all the symmetric representations are known for the $(2, 2p + 1)$ -torus knots [11] and the twist knots [10, 12, 13]. In addition, we have verified the results in [17, 18] for the colored HOMFLY polynomials of the $\mathbf{6}_2$, $\mathbf{6}_3$, $\mathbf{7}_3$, $\mathbf{7}_5$ knots up to 4 boxes. Hence, we present the [3]-colored HOMFLY polynomials for the other seven-crossing knots in §3.1. (The [2]-colored HOMFLY polynomials are collected in [14].) For each figure, we redraw the (left) diagram in the table of Rolfsen into the right **diagram** to which we apply the method in §2.

In §3.2, we shall compute the colored HOMFLY polynomials of thick knots [23]. If all the generators of the HOMFLY homology of a given knot have the same δ -grading, the knot is called homologically thin. (See more detail in [23].) Otherwise, it is called homologically thick. For a thick knot, the colored HOMFLY polynomial is a crucial information to obtain the homological invariant since **it is not clear the** homological invariant **obey** the exponential growth property. For the $(3, 4)$ -torus knot ($\mathbf{8}_{19}$) and the knot $\mathbf{9}_{42}$, the [2]-colored superpolynomials are given in [24]. In addition, the colored HOMFLY polynomials

of the knot $\mathbf{10}_{139}$ are given up to 4 boxes [17]. The evaluation of the colored HOMFLY polynomials for these knots are beyond the scope of the method provided by [10]. We have verified these results of knots $\mathbf{9}_{42}$ and $\mathbf{10}_{139}$ using our approach. Here we present the invariants for 10-crossing thick knots except the knot $\mathbf{10}_{161}$.

Before going into detail, let us fix the notation. In this paper, we use the skein relation

$$a^{1/2}P_{L_+} - a^{-1/2}P_{L_-} = (q^{1/2} - q^{-1/2})P_{L_0}, \quad (3.1)$$

for an uncolored HOMFLY polynomial $P(\mathcal{K}; a, q) = P_{[1]}(\mathcal{K}; a, q)$ with $P(\bigcirc; a, q) = 1$. To show the colored HOMFLY polynomials concisely, we use the following convention.

Example

$$\begin{aligned} & f(a, q) \begin{pmatrix} 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \end{pmatrix} \\ &= f(a, q) \left[(1 + 2q + 3q^2 + 4q^3) + a(5 + 6q + 7q^2 + 8q^3) + a^2(9 + 10q + 11q^2 + 12q^3) \right] \end{aligned} \quad (3.2)$$

In the matrix, the q -degree is assigned to the horizontal axis and the a -degree is scaled along the vertical axis.

3.1 Seven-crossing knots

3.1.1 7_4 knot

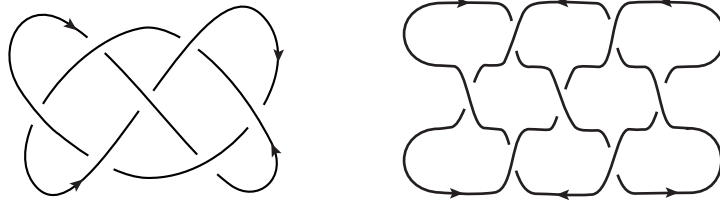


Figure 5. 7_4 knot

$$\begin{aligned} P_R(\mathbf{7}_4; a, q) &= \frac{1}{\dim_q R} \sum_{s,t,s',u,v} \epsilon_s^{R,R} \sqrt{\dim_q s} \epsilon_v^{R,R} \sqrt{\dim_q v} \hat{\lambda}_s^{(+)}(R, R) a_{ts} \begin{bmatrix} \bar{R} & R \\ R & \bar{R} \end{bmatrix} \\ & \quad (\hat{\lambda}_t^{(-)}(R, \bar{R}))^2 a_{ts'} \begin{bmatrix} R & \bar{R} \\ \bar{R} & R \end{bmatrix} \hat{\lambda}_{s'}^{(+)}(\bar{R}, \bar{R}) a_{us'} \begin{bmatrix} R & \bar{R} \\ \bar{R} & R \end{bmatrix} \\ & \quad (\hat{\lambda}_u^{(-)}(R, \bar{R}))^2 a_{uv} \begin{bmatrix} \bar{R} & R \\ R & \bar{R} \end{bmatrix} \hat{\lambda}_v^{(+)}(R, R). \end{aligned} \quad (3.3)$$

$$P_{[3]}(\mathbf{7}_4; a, q) = \frac{a^3}{q^3} \times$$

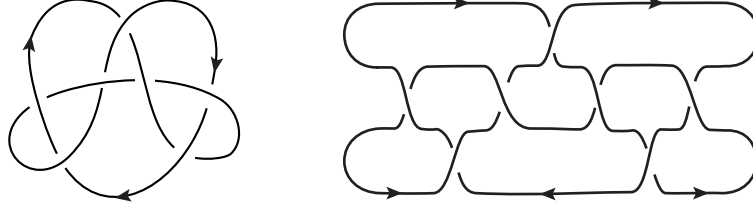


Figure 7. 7_7 knot

3.1.3 7_7 knot

$$\begin{aligned}
P_R(\mathbf{7}_7; a, q) &= \frac{1}{\dim_q R} \sum_{s,t,s',u,v,w,x} \epsilon_s^{R,R} \sqrt{\dim_q s} \epsilon_x^{R,R} \sqrt{\dim_q x} (\hat{\lambda}_s^{(+)}(R, R)) a_{ts} \begin{bmatrix} \bar{R} & R \\ R & \bar{R} \end{bmatrix} \\
&\quad (\hat{\lambda}_t^{(-)}(\bar{R}, R)) a_{ts'} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} (\hat{\lambda}_{s'}^{(-)}(\bar{R}, R))^{-1} a_{us'} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} \\
&\quad (\hat{\lambda}_u^{(+)}(\bar{R}, \bar{R}))^{-1} a_{uv} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} (\hat{\lambda}_v^{(-)}(R, \bar{R}))^{-1} a_{wv} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} \\
&\quad \hat{\lambda}_w^{(-)}(R, \bar{R}) a_{wx} \begin{bmatrix} \bar{R} & R \\ R & \bar{R} \end{bmatrix} \hat{\lambda}_x^{(+)}(R, R).
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
P_{[3]}(\mathbf{7}_7; a, q) &= \frac{1}{a^3 q^{13}} \times \\
&\quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 2 & -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & -4 & 2 & 6 & 8 & -3 & -4 & 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 4 & -6 & -12 & -2 & 11 & 9 & -13 & -18 & -3 & 7 & 5 & -6 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -4 & 0 & 6 & 13 & -8 & -19 & 2 & 30 & 22 & -16 & -25 & 4 & 20 & 13 & -7 & -4 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & -2 & 2 & 2 & -6 & -7 & 7 & 20 & -9 & -38 & -10 & 36 & 35 & -25 & -42 & -4 & 25 & 13 & -13 & -10 & 1 & 3 & 0 & -2 & 0 \\ 0 & 1 & -2 & 2 & 3 & -3 & -9 & 4 & 24 & 4 & -37 & -23 & 29 & 51 & -11 & -43 & -14 & 26 & 19 & -9 & -10 & 2 & 4 & 1 & -2 & 1 & 0 & 0 \\ -1 & 1 & 2 & -2 & -6 & 0 & 16 & 7 & -22 & -22 & 11 & 38 & 3 & -27 & -17 & 12 & 16 & -3 & -7 & -1 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -2 & -1 & 7 & 4 & -9 & -11 & 3 & 18 & 3 & -11 & -9 & 4 & 7 & -1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & -2 & -3 & 0 & 6 & 0 & -3 & -2 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{3.8}$$

3.2 Thick knots

3.2.1 10_{124} knot

We **note** that the knot $\mathbf{10}_{124}$ is the $(3, 5)$ -torus knot.

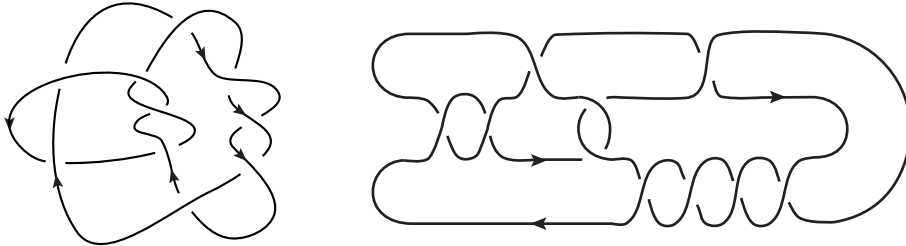


Figure 8. 10_{124} knot

$$P_R(\mathbf{10}_{124}; a, q) = \frac{1}{\dim_q R} \sum_{l,r,x,y,z} \frac{1}{\epsilon_l^{R,\bar{R}} \sqrt{\dim_q l}} \epsilon_r^{R,R} \sqrt{\dim_q r} \epsilon_x^{R,R} \sqrt{\dim_q x}$$

$$\begin{aligned}
& \times \epsilon_z(R, R) \sqrt{\dim_q z} a_{rl} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{xl} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{yl} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{zy} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} \\
& \times (\hat{\lambda}_r^{(+)}(R, R))^{-2} (\hat{\lambda}_x^{(+)}(R, R))^{-5} (\hat{\lambda}_y^{(-)}(R, \bar{R}))^{-1} (\hat{\lambda}_z^{(+)}(R, R))^{-2}
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
P_{[2]}(\mathbf{10}_{124}; a, q) &= \frac{1}{a^{12}q^{16}} \times \\
& \begin{pmatrix} 1 & 0 & 1 & 2 & 2 & 2 & 4 & 2 & 4 & 4 & 3 & 3 & 4 & 2 & 3 & 3 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ -1 & -2 & -2 & -4 & -6 & -6 & -8 & -9 & -8 & -9 & -9 & -8 & -8 & -7 & -5 & -5 & -5 & -3 & -2 & -2 & -1 & -1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 4 & 5 & 7 & 7 & 9 & 8 & 9 & 7 & 7 & 6 & 6 & 4 & 3 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & -2 & -3 & -4 & -3 & -3 & -4 & -3 & -2 & -2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{3.10}$$

3.2.2 10_{128} knot

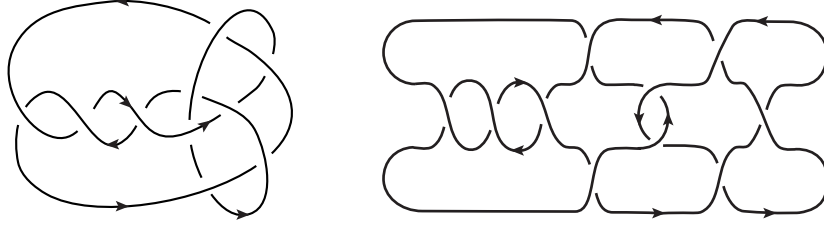


Figure 9. 10_{128} knot

$$\begin{aligned}
P_{\bar{R}}(\mathbf{10}_{128}; a, q) &= \sum_{l,r,x,y,x',y'} \frac{1}{\epsilon_l^{R,\bar{R}} \sqrt{\dim_q l}} \epsilon_r^{R,\bar{R}} \sqrt{\dim_q r} \epsilon_x^{R,\bar{R}} \sqrt{\dim_q x} \\
& \times \epsilon_{x'}^{R,\bar{R}} \sqrt{\dim_q x'} a_{rl} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{yl} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{y'l} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{yx} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} \\
& \times a_{y'x'} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} (\hat{\lambda}_r^{(-)}(R, \bar{R}))^{-2} (\hat{\lambda}_y^{(+)}(R, R))^{-2} (\hat{\lambda}_x^{(-)}(R, \bar{R}))^{-1} \\
& \times (\hat{\lambda}_{y'}^{(+)}(R, R))^{-2} (\hat{\lambda}_{x'}^{(-)}(R, \bar{R}))^{-3}
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
P_{[2]}(\mathbf{10}_{128}; a, q) &= \frac{1}{a^{12}q^{14}} \times \\
& \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 2 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 2 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 2 & 1 & -2 & -1 & 3 & 1 & -1 & 1 & 1 & 1 & 0 & 0 & 3 & 0 & -2 & 1 & 1 \\ 1 & -1 & -1 & 2 & 1 & -2 & -2 & 1 & 1 & -3 & -2 & -1 & -1 & -2 & -1 & 1 & -1 & -2 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & -3 & -4 & 1 & 1 & -3 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 3 & 4 & 1 & 1 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{3.12}$$

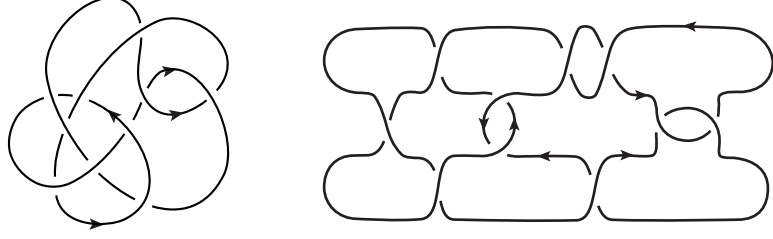


Figure 10. 10_{132} knot

3.2.3 10_{132} knot

$$\begin{aligned}
P_R(\mathbf{10}_{132}; a, q) = & \sum_{l,r,x,y,x',y'} \frac{1}{\epsilon_l^{R,\bar{R}} \sqrt{\dim_q l}} \epsilon_r^{R,R} \sqrt{\dim_q r} \epsilon_x^{R,R} \sqrt{\dim_q x} \\
& \times \epsilon_{x'}^{R,\bar{R}} \sqrt{\dim_q x'} a_{rl} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{yl} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{y'l} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{xy} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} \\
& \times a_{y'x'} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} (\hat{\lambda}_r^{(+)}(R, R))^2 (\hat{\lambda}_y^{(-)}(R, \bar{R}))^3 (\hat{\lambda}_x^{(+)}(R, R))^2 \\
& \times (\hat{\lambda}_{y'}^{(+)}(R, R))^{-2} (\hat{\lambda}_{x'}^{(-)}(R, \bar{R}))^{-1}
\end{aligned} \tag{3.13}$$

$$P_{[2]}(\mathbf{10}_{132}; a, q) = \frac{a}{q^4} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -2 & -1 & -2 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 & 1 & -1 & 2 & 1 & 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 2 & 2 & -2 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 & 2 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.14}$$

3.2.4 10_{136} knot

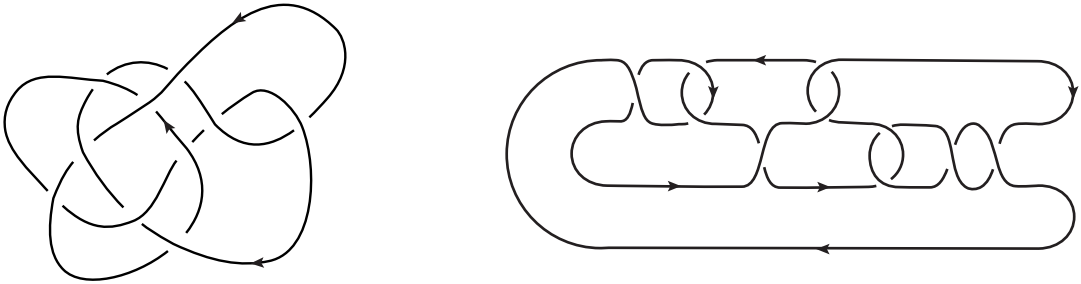


Figure 11. 10_{136} knot

$$\begin{aligned}
P_R(\mathbf{10}_{136}; a, q) &= \sum_{l, l_1, r, x, y, z} \frac{1}{\epsilon_l^{R, \bar{R}} \sqrt{\dim_q l}} \epsilon_x^{R, R} \sqrt{\dim_q x} \epsilon_r^{R, \bar{R}} \sqrt{\dim_q r} \\
&\times \epsilon_z^{R, \bar{R}} \sqrt{\dim_q u} a_{xl} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{rl_1} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{ll_1} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{yl} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} \\
&\times a_{yz} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} (\hat{\lambda}_x^{(+)}(R, R))^{-2} (\hat{\lambda}_r^{(-)}(R, \bar{R}))^{-2} (\hat{\lambda}_{l_1}^{(-)}(R, \bar{R}))^2 \\
&\times (\hat{\lambda}_y^{(+)}(R, R))^2 (\hat{\lambda}_z^{(-)}(R, \bar{R}))^{-3}
\end{aligned} \tag{3.15}$$

$$P_{[2]}(\mathbf{10}_{136}; a, q) = \frac{1}{a^4 q^6} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -2 & -1 & 2 & -2 & -2 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 3 & 0 & -1 & 1 \\ -1 & 0 & -2 & -3 & 1 & -1 & -4 & -2 & 0 & 0 & -1 & -1 & 0 & 0 \\ 2 & 0 & 0 & 4 & 4 & -1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 & -3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.16}$$

3.2.5 10_{145} knot

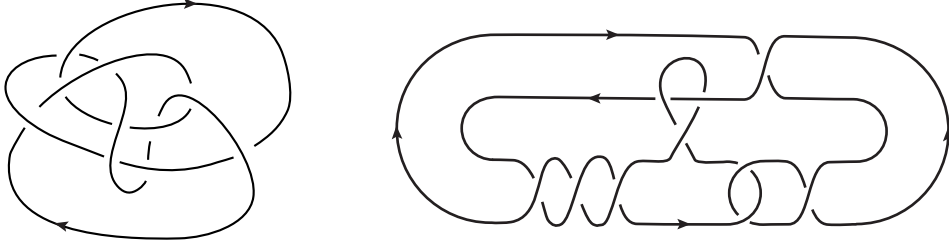


Figure 12. 10_{145} knot

$$\begin{aligned}
P_R(\mathbf{10}_{145}; a, q) &= \sum_{l, r, u, v, x, y, z} \frac{1}{\epsilon_l^{R, R} \sqrt{\dim_q l}} \epsilon_z^{R, R} \sqrt{\dim_q z} \epsilon_u^{R, \bar{R}} \sqrt{\dim_q u} \\
&\times \epsilon_r^{R, R} \sqrt{\dim_q r} a_{lx} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{yx} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{zy} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{lu} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} \\
&\times a_{lv} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{rv} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} (\hat{\lambda}_x^{(-)}(R, \bar{R}))^{-1} \hat{\lambda}_y^{(-)}(R, \bar{R}) \hat{\lambda}_z^{(+)}(R, R) \\
&\times (\hat{\lambda}_u^{(-)}(R, \bar{R}))^3 (\hat{\lambda}_v^{(-)}(R, \bar{R}))^2 (\hat{\lambda}_r^{(+)}(R, R))^2
\end{aligned} \tag{3.17}$$

$$P_{[2]}(\mathbf{10}_{145}; a, q) = \frac{a^4}{q^4} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -2 & 1 & 0 & -3 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 2 & 2 & -2 & 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 1 & 2 & -3 & 0 & 1 & -2 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 & -1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad (3.18)$$

3.2.6 10_{152} knot

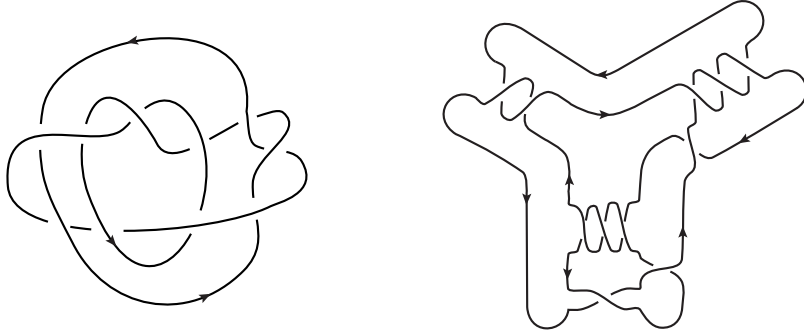


Figure 13. 10_{152} knot

$$\begin{aligned} P_R(\mathbf{10}_{152}; a, q) = & \sum_{l, l_1, r, u, v, x, y, z} \frac{1}{\epsilon_l^{R, \bar{R}} \sqrt{\dim_q l} \epsilon_{l_1}^{R, \bar{R}} \sqrt{\dim_q l_1}} \epsilon_z^{R, \bar{R}} \sqrt{\dim_q z} \\ & \times \epsilon_u^{R, \bar{R}} \sqrt{\dim_q u} \epsilon_r^{R, R} \sqrt{\dim_q r} \epsilon_v^{R, R} \sqrt{\dim_q v} \\ & \times a_{rl_1} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{ll_1} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{yl_1} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{yz} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{lu} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} \\ & \times a_{vl} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} (\hat{\lambda}_r^{(+)}(R, R))^3 \hat{\lambda}_y^{(+)}(R, R) \hat{\lambda}_z^{(-)}(R, \bar{R}) \\ & \times (\hat{\lambda}_u^{(-)}(R, \bar{R}))^2 \hat{\lambda}_l^{(-)}(R, \bar{R}) (\hat{\lambda}_v^{(+)}(R, R))^3 \end{aligned} \quad (3.19)$$

$$P_{[2]}(\mathbf{10}_{152}; a, q) = \frac{1}{a^{12} q^{16}} \begin{pmatrix} 1 & 0 & 1 & 3 & 2 & 2 & 6 & 2 & 6 & 5 & 2 & 7 & 6 & 0 & 5 & 6 & 2 & 0 & 3 & 2 & 1 & 1 & 0 & 0 & 1 \\ -1 & -3 & -2 & -5 & -9 & -8 & -11 & -11 & -13 & -16 & -9 & -11 & -17 & -10 & -4 & -8 & -10 & -4 & -1 & -3 & -2 & -1 & -1 & 0 & 0 \\ 0 & 3 & 3 & 6 & 7 & 10 & 14 & 12 & 9 & 20 & 13 & 6 & 12 & 13 & 6 & 4 & 3 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -4 & -3 & -3 & -9 & -8 & -1 & -7 & -10 & -3 & -2 & -3 & -3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 & 0 & 5 & 0 & -2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.20)$$

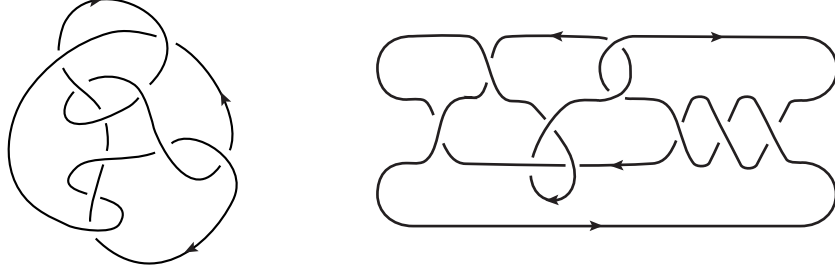


Figure 14. 10_{153} knot

3.2.7 10_{153} knot

$$\begin{aligned}
P_R(\mathbf{10}_{153}; a, q) = & \sum_{l, l_1, v, u, x, y, z, r} \frac{1}{\epsilon_l^{R, \bar{R}} \sqrt{\dim_q l} \epsilon_{l_1}^{R, \bar{R}} \sqrt{\dim_q l_1}} \epsilon_z^{R, \bar{R}} \sqrt{\dim_q z} \\
& \times \epsilon_v^{R, \bar{R}} \sqrt{\dim_q v} \epsilon_u^{R, R} \sqrt{\dim_q u} \epsilon_r^{R, R} \sqrt{\dim_q r} \\
& \times a_{lx} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{yx} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{yz} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{ll_1} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{lv} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} \\
& \times a_{rl_1} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{ul_1} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} (\hat{\lambda}_x^{(-)}(R, \bar{R}))^{-1} (\hat{\lambda}_y^{(+)}(R, R))^{-1} (\hat{\lambda}_z^{(-)}(R, \bar{R}))^{-1} \\
& \times (\hat{\lambda}_r^{(+)}(R, R))^2 (\hat{\lambda}_v^{(-)}(R, \bar{R}))^{-2} (\hat{\lambda}_u^{(+)}(R, R))^3
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
P_{[2]}(\mathbf{10}_{153}; a, q) = & \frac{1}{a^4 q^9} \\
& \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -2 & -5 & -2 & -2 & -6 & -3 & -2 & -4 & -2 & -2 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 3 & 2 & -1 & 5 & 5 & 3 & 5 & 6 & 2 & 3 & 3 & 2 & 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 3 & 2 & -3 & 1 & 2 & -4 & 0 & 1 & -4 & -1 & 0 & -2 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -2 & -2 & 0 & -2 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{3.22}$$

3.2.8 10_{154} knot

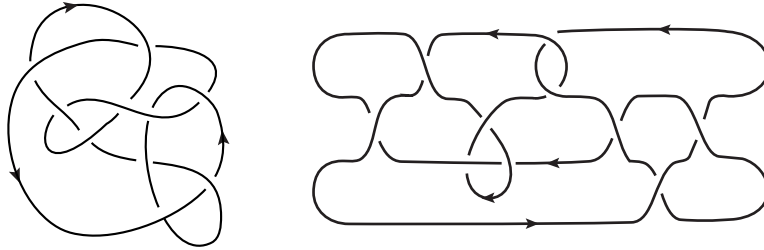


Figure 15. 10_{154} knot

$$\begin{aligned}
P_R(\mathbf{10}_{154}; a, q) = & \sum_{l, l_1, r, s, u, v, w, x, y, z} \frac{1}{\epsilon_l^{R, \bar{R}} \sqrt{\dim_q l} \epsilon_{l_1}^{R, \bar{R}} \sqrt{\dim_q l_1}} \epsilon_z^{R, \bar{R}} \sqrt{\dim_q z} \\
& \times \epsilon_s^{R, \bar{R}} \sqrt{\dim_q s} \epsilon_w^{R, \bar{R}} \sqrt{\dim_q w} \epsilon_r^{R, \bar{R}} \sqrt{\dim_q r} \\
& \times a_{lx} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{yx} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{yz} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{ll_1} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{ls} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} \\
& \times a_{rl_1} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{l_1 u} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} a_{vu} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} a_{vw} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} (\hat{\lambda}_x^{(-)}(R, \bar{R}))^{-1} \\
& \times (\hat{\lambda}_y^{(+)}(R, R))^{-1} (\hat{\lambda}_z^{(-)}(R, \bar{R}))^{-1} (\hat{\lambda}_r^{(-)}(R, \bar{R}))^{-2} (\hat{\lambda}_s^{(-)}(R, \bar{R}))^{-2} \\
& \times (\hat{\lambda}_u^{(-)}(R, \bar{R}))^{-1} (\hat{\lambda}_v^{(+)}(R, R))^{-1} (\hat{\lambda}_w^{(-)}(R, \bar{R}))^{-1}
\end{aligned} \tag{3.23}$$

$$P_{[2]}(\mathbf{10}_{154}; a, q) = \frac{a^6}{q^6} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -4 & 2 & 4 & -3 & -2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 1 & 4 & 5 & -1 & -2 & 3 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -2 & 0 & 3 & -2 & -6 & -1 & 2 & -1 & -2 & -2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -3 & -1 & 5 & -2 & -7 & 1 & 2 & -3 & -1 & -2 & -3 & 1 & -1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 3 & 2 & -3 & 2 & 4 & -1 & 1 & 3 & -1 & 2 & 2 & 0 & 0 & 1 \end{pmatrix} \tag{3.24}$$

4 Colored HOMFLY invariants for links

In this section, we shall compute the colored HOMFLY invariants of links. First of all, we should emphasize that the invariants are no longer polynomials, but *rational functions* with respect to the variables (a, q) . In addition, the colored HOMFLY invariants of links are crucially dependent of the orientation of each component of a link. For each link in this section, we choose the orientation presented in Knot Atlas [25]. In [16], the cyclotomic expansions of the colored HOMFLY invariants of the twist links including the Whitehead link 5_1^2 and the link 7_3^2 , and the Borromean rings 6_3^3 are given. Therefore, we treat two-component links with six and seven crossings in §4.1. We have not succeeded to compute the link 7_6^2 in this method. In §4.2, we consider three-component links including the (3,3)-torus link and the link 6_1^3 .

For two-component links, the colored HOMFLY invariants are symmetric under interchanging the two colors $\bar{P}_{([n_1], [n_2])}(\mathcal{L}; a, q) = \bar{P}_{([n_2], [n_1])}(\mathcal{L}; a, q)$. For two-component links §4.1, an unreduced colored HOMFLY invariant $\bar{P}_{([n_1], [n_2])}(\mathcal{L}; a, q)$ contains the unknot factor $\bar{P}_{[n_{\max}]}(\bigcirc; a, q)$ colored by the highest rank $n_{\max} = \max(n_1, n_2)$. Furthermore, we observed that it includes the factor $(a; q)_{n_{\max}} / (q; q)_{n_1} (q; q)_{n_2}$. If we normalize by

$$\frac{(q; q)_{n_1} (q; q)_{n_2}}{(a; q)_{n_{\max}}} \bar{P}_{([n_1], [n_2])}(\mathcal{L}; a, q), \tag{4.1}$$

then it becomes a Laurent polynomial with respect to the variables (a, q) . Interestingly, it satisfies the exponential growth property (the **property special** polynomials satisfy) [26–28]

$$\lim_{q \rightarrow 1} \frac{(q; q)_{kn_1} (q; q)_{kn_2}}{(a; q)_{kn_{\max}}} \bar{P}_{([kn_1], [kn_2])}(\mathcal{L}; a, q) = \left[\lim_{q \rightarrow 1} \frac{(q; q)_{n_1} (q; q)_{n_2}}{(a; q)_{n_{\max}}} \bar{P}_{([n_1], [n_2])}(\mathcal{L}; a, q) \right]^k. \quad (4.2)$$

where $\gcd(n_1, n_2) = 1$ and $k \in \mathbb{Z}_{\geq 0}$. In fact, the form of the Laurent polynomial (5.2) strongly **suggest** the interpretation at homological level. For instance, it is easy to see that the difference between the $([1], [3])$ -color invariant and the $([1], [4])$ -color invariant in the matrix form expressions below is just a shift in q -degree. In higher ranks, though the cancellation between coefficients make this shift obscure, it is not difficult that only a shift in q -degree is involved if you increase the rank of the **larger** color. The homological interpretations of link invariants will be given in the separate paper [16].

Similarly, for three-component links, **an** colored HOMFLY invariant $\bar{P}_{([n_1], [n_2], [n_3])}(\mathcal{L}; a, q)$ contains the unknot factor $\bar{P}_{[n_{\max}]}(\bigcirc; a, q)$ colored by the highest rank $n_{\max} = \max(n_1, n_2, n_3)$. In addition, it also includes the factor $(a; q)_{n_{\max}} / (q; q)_{n_1} (q; q)_{n_2} (q; q)_{n_3}$. If we normalize by

$$\frac{(q; q)_{n_1} (q; q)_{n_2} (q; q)_{n_3}}{(a; q)_{n_{\max}}} \bar{P}_{([n_1], [n_2], [n_3])}(\mathcal{L}; a, q), \quad (4.3)$$

then it becomes a Laurent polynomial, which obeys the exponential growth property

$$\begin{aligned} & \lim_{q \rightarrow 1} \frac{(q; q)_{kn_1} (q; q)_{kn_2} (q; q)_{kn_3}}{(a; q)_{kn_{\max}}} \bar{P}_{([kn_1], [kn_2], [kn_3])}(\mathcal{L}; a, q) \\ &= \left[\lim_{q \rightarrow 1} \frac{(q; q)_{n_1} (q; q)_{n_2} (q; q)_{n_3}}{(a; q)_{n_{\max}}} \bar{P}_{([n_1], [n_2], [n_3])}(\mathcal{L}; a, q) \right]^k \end{aligned} \quad (4.4)$$

where $\gcd(n_1, n_2, n_3) = 1$.

4.1 Two-component links

4.1.1 6_2^2 link

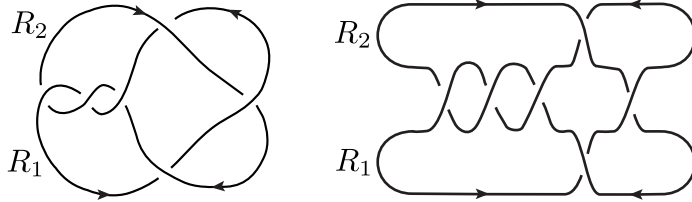


Figure 16. 6_2^2 link

$$\begin{aligned} \bar{P}_{(R_1, R_2)}(\mathbf{6}_2^2; a, q) &= q^{\frac{3\ell^{(1)}\ell^{(2)}}{N}} \sum_{s, t, s'} \epsilon_s^{R_1, R_2} \sqrt{\dim_q s} \epsilon_{s'}^{\bar{R}_1, \bar{R}_2} \sqrt{\dim_q s'} (\lambda_s^{(+)}(R_1, R_2))^{-3} \\ &\times a_{t\bar{s}} \begin{bmatrix} R_2 & \bar{R}_1 \\ R_2 & R_1 \end{bmatrix} (\lambda_t^{(-)}(\bar{R}_1, R_2))^{-2} a_{t\bar{s}'} \begin{bmatrix} \bar{R}_1 & R_2 \\ R_1 & \bar{R}_2 \end{bmatrix} (\lambda_{s'}^{(+)}(\bar{R}_1, \bar{R}_2))^{-1}. \end{aligned} \quad (4.5)$$

- $\bar{P}_{([1],[1])}(\mathbf{6}_2^2; a, q) =$

$$\frac{(1-a)}{a^4 q (1-q)^2} \begin{pmatrix} -1 & 2 & -2 & 2 & -1 \\ -1 & 2 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix} \quad (4.6)$$

- $\bar{P}_{([1],[2])}(\mathbf{6}_2^2; a, q) =$

$$\frac{(1-a)(1-aq)}{a^{9/2} q^{7/2} (1-q)^2 (1-q^2)} \begin{pmatrix} 0 & -1 & 1 & 1 & -2 & 1 & 1 & -1 \\ -1 & 0 & 2 & -2 & -1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (4.7)$$

- $\bar{P}_{([1],[3])}(\mathbf{6}_2^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^5 q^6 (1-q)^2 (1-q^2) (1-q^3)} \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 2 & -2 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.8)$$

- $\bar{P}_{([1],[4])}(\mathbf{6}_2^2; a, q) =$

$$\frac{q^{\frac{1}{2}} (1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^{11/2} q^{18/2} (1-q)^2 (1-q^2) (1-q^3) (1-q^4)} \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.9)$$

- $\bar{P}_{([2],[2])}(\mathbf{6}_2^2; a, q) =$

$$\frac{(1-a)(1-aq)}{a^8 q^7 (1-q)^2 (1-q^2)^2} \begin{pmatrix} 0 & 0 & 1 & -2 & 0 & 3 & -4 & 1 & 5 & -5 & -2 & 5 & -1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 5 & -6 & -2 & 10 & -5 & -7 & 7 & 2 & -4 & 0 & 1 \\ 1 & -2 & 0 & 5 & -5 & -3 & 10 & -3 & -7 & 5 & 2 & -2 & 0 & 0 & 0 \\ -1 & 0 & 2 & -3 & -2 & 5 & -1 & -4 & 2 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.10)$$

- $\bar{P}_{([2],[3])}(\mathbf{6}_2^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^{17/2} q^{25/2} (1-q)^2 (1-q^2)^2 (1-q^3)} \times$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & -2 & 2 & 2 & -2 & -3 & 2 & 5 & -2 & -5 & 1 & 3 & 1 & -2 & -1 & 1 \\ 0 & 0 & 1 & 0 & -3 & 0 & 5 & 1 & -7 & -3 & 8 & 5 & -7 & -6 & 4 & 5 & -1 & -3 & 0 & 1 & 0 \\ 1 & 0 & -2 & -1 & 3 & 4 & -4 & -6 & 4 & 7 & -1 & -7 & -1 & 5 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 2 & -1 & -4 & 0 & 4 & 1 & -3 & -2 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.11)$$

- $\bar{P}_{([2],[4])}(\mathbf{6}_2^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^9 q^{18} (1-q)^2 (1-q^2)^2 (1-q^3)(1-q^4)} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & 2 & -2 & -1 & -1 & 1 & 4 & -1 & -2 & -2 & 0 & 3 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -2 & 0 & 4 & 1 & -1 & -5 & -2 & 5 & 3 & 2 & -5 & -5 & 3 & 2 & 2 & 0 & -3 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & -1 & 2 & 2 & 3 & -4 & -4 & 1 & 2 & 6 & -2 & -4 & -1 & -1 & 4 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 2 & -1 & -2 & -2 & 0 & 4 & 0 & -1 & -1 & -2 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.12)$$

4.1.2 6_3^2 link

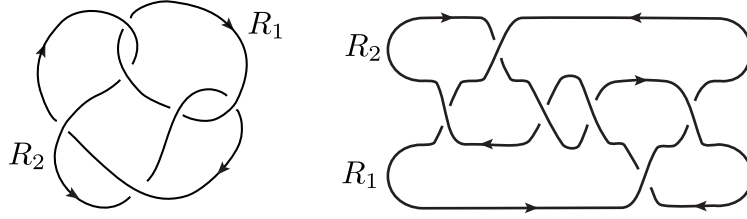


Figure 17. 6_3^2 link

$$\begin{aligned} \bar{P}_{(R_1, R_2)}(\mathbf{6}_3^2; a, q) &= q^{(-2C_{R_2} - \frac{2\ell^{(1)}\ell^{(2)}}{N})} \sum_{s, t, s', u, v} \epsilon_s^{R_1, R_2} \sqrt{\dim_q s} \epsilon_v^{\bar{R}_1, \bar{R}_2} \sqrt{\dim_q v} \lambda_s^{(+)}(R_1, R_2) \\ &\quad a_{ts} \begin{bmatrix} \bar{R}_1 & R_2 \\ R_1 & \bar{R}_2 \end{bmatrix} \lambda_t^{(-)}(R_1, \bar{R}_2) a_{ts'} \begin{bmatrix} \bar{R}_1 & R_2 \\ \bar{R}_2 & R_1 \end{bmatrix} (\lambda_{s'}^{(-)}(R_2, \bar{R}_2))^{-2} \\ &\quad a_{us'} \begin{bmatrix} \bar{R}_1 & R_2 \\ \bar{R}_2 & R_1 \end{bmatrix} \lambda_u^{(-)}(\bar{R}_1, R_2) a_{uv} \begin{bmatrix} R_2 & \bar{R}_1 \\ \bar{R}_2 & R_1 \end{bmatrix} \lambda_v^{(+)}(\bar{R}_1, \bar{R}_2). \end{aligned} \quad (4.13)$$

- $\bar{P}_{([1],[1])}(\mathbf{6}_3^2; a, q) =$

$$\frac{(1-a)}{aq(1-q)^2} \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & -3 & 2 & 0 \\ -1 & 3 & -4 & 3 & -1 \\ 0 & 1 & -2 & 1 & 0 \end{pmatrix} \quad (4.14)$$

- $\bar{P}_{([1],[2])}(\mathbf{6}_3^2; a, q) =$

$$\frac{(1-a)(1-aq)}{a^{3/2} q^{1/2} (1-q)^2 (1-q^2)} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 & -1 & 2 & 0 \\ -1 & 2 & 0 & -3 & 2 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix} \quad (4.15)$$

- $\bar{P}_{([1],[3])}(\mathbf{6}_3^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^2(1-q)^2(1-q^2)(1-q^3)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & -1 & 2 & 0 \\ -1 & 2 & -1 & 1 & -3 & 2 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(4.16)

- $\bar{P}_{([1],[4])}(\mathbf{6}_3^2; a, q) =$

$$\frac{q^{1/2}(1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^{5/2}(1-q)^2(1-q^2)(1-q^3)(1-q^4)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & -1 & 2 & 0 \\ -1 & 2 & -1 & 0 & 1 & -3 & 2 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(4.17)

- $\bar{P}_{([2],[2])}(\mathbf{6}_3^2; a, q) =$

$$\frac{(1-a)(1-aq)}{a^2q^3(1-q)^2(1-q^2)^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 3 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -6 & -1 & 10 & -3 & -6 & 3 & 1 \\ 0 & 0 & -2 & 3 & 5 & -12 & -2 & 17 & -5 & -10 & 6 & 2 & -2 \\ 1 & -3 & 1 & 9 & -11 & -8 & 19 & -2 & -13 & 7 & 2 & -3 & 1 \\ -1 & 1 & 4 & -5 & -5 & 9 & 1 & -7 & 2 & 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 4 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(4.18)

- $\bar{P}_{([2],[3])}(\mathbf{6}_3^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^{5/2}q^{5/2}(1-q)^2(1-q^2)^2(1-q^3)} \times$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -4 & -4 & 4 & 6 & 0 & -6 & -2 & 3 & 1 \\ 0 & 0 & 0 & -2 & 2 & 5 & -3 & -9 & -1 & 12 & 6 & -9 & -7 & 3 & 5 & 0 & -2 \\ 1 & -2 & -2 & 6 & 4 & -8 & -10 & 6 & 15 & -2 & -12 & -2 & 6 & 3 & -3 & -1 & 1 \\ -1 & 0 & 4 & 1 & -7 & -4 & 7 & 7 & -4 & -7 & 1 & 4 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 2 & 1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(4.19)

- $\bar{P}_{([2],[4])}(\mathbf{6}_3^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^3q^2(1-q)^2(1-q^2)^2(1-q^3)(1-q^4)} \times$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -4 & -2 & 1 & 1 & 6 & -1 & -3 & -2 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 & -2 & 2 & 4 & -3 & -1 & -5 & -2 & 9 & 3 & 1 & -5 & -6 & 2 & 2 & 3 & 0 & -2 \\ 1 & -2 & -1 & 3 & 1 & 3 & -5 & -7 & 2 & 3 & 9 & 0 & -7 & -3 & -2 & 5 & 2 & -1 & -1 & -1 & 1 \\ -1 & 0 & 3 & 1 & -2 & -4 & -2 & 4 & 4 & 2 & -3 & -5 & 0 & 2 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(4.20)

- $\overline{P}_{([1],[3])}(\mathbf{7}_1^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^4 q^5 (1-q)^2 (1-q^2)(1-q^3)} \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 & 3 & -1 & 0 & -2 & 3 & -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \quad (4.25)$$

- $\overline{P}_{([1],[4])}(\mathbf{7}_1^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^9 q^{13/2} (1-q)^2 (1-q^2)(1-q^3)(1-q^4)} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & -2 & 3 & -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.26)$$

- $\overline{P}_{([2],[2])}(\mathbf{7}_1^2; a, q) =$

$$\frac{(1-a)(1-aq)}{a^6 q^8 (1-q)^2 (1-q^2)^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & 1 & -2 & 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & -3 & 1 & 1 & -3 & 3 & 1 & -6 & 2 & 5 & -4 & -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 3 & -5 & 2 & 3 & -7 & 6 & 5 & -12 & 2 & 11 & -7 & -5 & 6 & 0 & -2 & 1 \\ -1 & 1 & 2 & -4 & 2 & 3 & -8 & 5 & 7 & -11 & -1 & 10 & -3 & -5 & 3 & 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 3 & -4 & 1 & 5 & -5 & -2 & 5 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.27)$$

- $\overline{P}_{([2],[3])}(\mathbf{7}_1^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^{13/2} q^{23/2} (1-q)^2 (1-q^2)^2 (1-q^3)} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 2 & 0 & -2 & 0 & 1 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & -3 & -1 & 3 & 1 & -4 & -1 & 5 & 1 & -5 & -3 & 4 & 4 & -3 & -3 & 1 & 2 & 0 & -1 & 0 \\ 1 & -1 & -2 & 3 & 3 & -4 & -4 & 4 & 6 & -6 & -7 & 8 & 8 & -6 & -10 & 3 & 11 & -1 & -8 & -1 & 4 & 2 & -2 & -1 & 1 & 0 \\ -1 & 0 & 3 & 0 & -5 & 0 & 7 & 0 & -9 & -1 & 11 & 3 & -11 & -5 & 8 & 6 & -4 & -5 & 1 & 3 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 2 & 2 & -2 & -3 & 2 & 5 & -2 & -5 & 1 & 3 & 1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.28)$$

- $\overline{P}_{([2],[4])}(\mathbf{7}_1^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^7 q^{15} (1-q)^2 (1-q^2)^2 (1-q^3)(1-q^4)} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -2 & 0 & 1 & 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -3 & -1 & 1 & 2 & 1 & -3 & -2 & 1 & 4 & 1 & -4 & -2 & -1 & 3 & 4 & -3 & -2 & 0 & 0 & 2 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 & 2 & 3 & -3 & -3 & -1 & 3 & 5 & -2 & -5 & -4 & 6 & 7 & -2 & -4 & -7 & 2 & 9 & 0 & -2 & -4 & -2 & 4 & 1 & 0 & -1 & -1 & 1 & 0 \\ -1 & 0 & 2 & 1 & -1 & -4 & 0 & 4 & 3 & -1 & -7 & -1 & 4 & 6 & 2 & -9 & -4 & 3 & 4 & 4 & -3 & -4 & 0 & 1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 & 2 & -2 & -1 & -1 & 1 & 4 & -1 & -2 & -2 & 0 & 3 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.29)$$

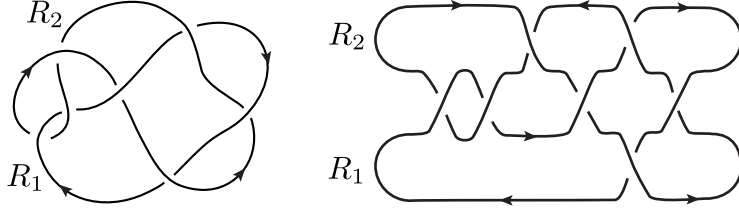


Figure 19. 7_2^2 link

4.1.4 7_2^2 link

$$\begin{aligned}
\bar{P}_{(R_1, R_2)}(\mathbf{7}_2^2; a, q) = & q^{\left(-C_{R_2} + \frac{\ell^{(1)}\ell^{(2)}}{N}\right)} \sum_{s, t, s', u, v} \epsilon_s^{\bar{R}_1, R_2} \sqrt{\dim_q s} \epsilon_v^{R_1, R_2} \sqrt{\dim_q v} (\lambda_s^{(-)}(\bar{R}_1, R_2))^2 \\
& \times a_{ts} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} (\lambda_t^{(-)}(R_2, \bar{R}_2))^{-1} a_{s't} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_2 & \bar{R}_1 \end{bmatrix} (\lambda_{s'}^{(+)}(R_1, R_2))^{-1} \\
& \times a_{s'u} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_1 & \bar{R}_2 \end{bmatrix} (\lambda_u^{(-)}(\bar{R}_1, R_2))^{-2} a_{vu} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_1 & \bar{R}_2 \end{bmatrix} (\lambda_v^{(+)}(R_1, R_2))^{-1}
\end{aligned} \tag{4.30}$$

$$\bullet \bar{P}_{([1], [1])}(\mathbf{7}_2^2; a, q) =$$

$$\frac{(1-a)}{a^3 q (1-q)^2} \begin{pmatrix} 0 & -1 & 2 & -1 & 0 \\ 1 & -4 & 5 & -4 & 1 \\ 1 & -3 & 5 & -3 & 1 \\ 0 & -1 & 2 & -1 & 0 \end{pmatrix} \tag{4.31}$$

$$\bullet \bar{P}_{([1], [2])}(\mathbf{7}_2^2; a, q) =$$

$$\frac{(1-a)(1-aq)}{a^{7/2} q^{5/2} (1-q)^2 (1-q^2)} \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & 0 & 4 & -3 & -1 & 1 \\ 1 & -1 & -2 & 4 & 0 & -2 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.32}$$

$$\bullet \bar{P}_{([1], [3])}(\mathbf{7}_2^2; a, q) =$$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^4 q^4 (1-q)^2 (1-q^2) (1-q^3)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 & 1 & -1 & 4 & -3 & 0 & -1 & 1 \\ 1 & -1 & 0 & -2 & 4 & -1 & 1 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.33}$$

$$\bullet \bar{P}_{([1], [4])}(\mathbf{7}_2^2; a, q) =$$

$$\frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^{9/2} q^{11/2} (1-q)^2 (1-q^2) (1-q^3) (1-q^4)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 0 & -1 & 4 & -3 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & -2 & 4 & -1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(4.34)

$$\bullet \overline{P}_{([2],[2])}(\mathbf{7}_2^2; a, q) =$$

$$\frac{(1-a)(1-aq)}{a^6 q^6 (1-q)^2 (1-q^2)^2} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & -1 & 4 & -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 3 & -8 & -1 & 12 & -4 & -8 & 4 & 2 & -1 & \\ 0 & 0 & 1 & -4 & 2 & 12 & -17 & -8 & 29 & -6 & -20 & 11 & 5 & -4 & 0 & \\ 0 & 1 & -4 & 1 & 13 & -15 & -15 & 29 & 2 & -25 & 7 & 9 & -5 & -1 & 1 & \\ 1 & -3 & 1 & 10 & -11 & -11 & 21 & 2 & -16 & 4 & 5 & -2 & 0 & 0 & 0 & \\ -1 & 1 & 4 & -6 & -5 & 11 & 1 & -8 & 2 & 2 & -1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & -2 & -1 & 4 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

(4.35)

$$\bullet \overline{P}_{([2],[3])}(\mathbf{7}_2^2; a, q) =$$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^{13/2} q^{19/2} (1-q)^2 (1-q^2)^2 (1-q^3)} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 1 & 2 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 4 & -2 & -7 & -1 & 9 & 5 & -7 & -6 & 2 & 4 & 0 & -1 & \\ 0 & 0 & 0 & 0 & 1 & -3 & -2 & 10 & 4 & -14 & -12 & 13 & 21 & -7 & -19 & 0 & 10 & 4 & -4 & -2 & 1 & \\ 0 & 0 & 1 & -2 & -4 & 5 & 11 & -6 & -21 & -1 & 25 & 10 & -19 & -14 & 8 & 10 & -2 & -4 & 0 & 1 & 0 & \\ 1 & -1 & -3 & 2 & 7 & 2 & -13 & -9 & 14 & 13 & -5 & -13 & -1 & 8 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & \\ -1 & 0 & 3 & 2 & -5 & -6 & 4 & 8 & 0 & -6 & -2 & 3 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & -1 & -2 & 1 & 2 & 1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

(4.36)

$$\bullet \overline{P}_{([2],[4])}(\mathbf{7}_2^2; a, q) =$$

$$\frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^7 q^{13} (1-q)^2 (1-q^2)^2 (1-q^3)(1-q^4)} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & -1 & -2 & -4 & -1 & 6 & 3 & 1 & -4 & -5 & 1 & 2 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & -1 & 6 & 2 & 2 & -10 & -8 & 6 & 6 & 13 & -4 & -13 & -2 & 0 & 7 & 3 & -2 & -2 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & -2 & 1 & 3 & 8 & -5 & -11 & -5 & 0 & 17 & 7 & -7 & -8 & -9 & 5 & 7 & 1 & -1 & -3 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 & 1 & 6 & 0 & -2 & -8 & -6 & 10 & 6 & 4 & -3 & -9 & -1 & 2 & 4 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 & 1 & -4 & -4 & 2 & 2 & 5 & 0 & -4 & -1 & -1 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(4.37)

4.1.5 7_4^2 link

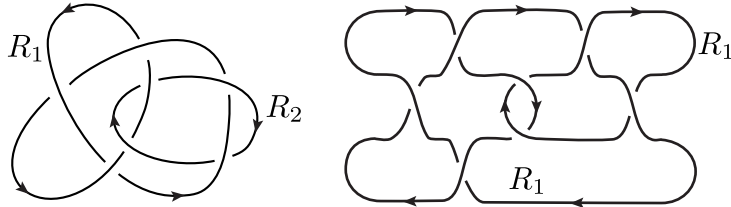


Figure 20. 7_4^2 link

$$\begin{aligned}
\bar{P}_{R_1, R_2}(\mathbf{7}_4^2; a, q) &= q^{-3C_{R_1}} \sum_{l, r, u, v, x, y} \frac{1}{\epsilon_l^{R, \bar{R}} \sqrt{\dim_q l}} \epsilon_r^{R_1, R_2} \sqrt{\dim_q r} \epsilon_u^{R_1, \bar{R}_1} \\
&\times \sqrt{\dim_q u} \epsilon_v^{R_1, R_2} \sqrt{\dim_q v} a_{rl} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_2 & \bar{R}_1 \end{bmatrix} a_{xl} \begin{bmatrix} R_1 & R_1 \\ \bar{R}_1 & \bar{R}_1 \end{bmatrix} a_{ly} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} \\
&\times a_{xu} \begin{bmatrix} R_1 & R_1 \\ \bar{R}_1 & \bar{R}_1 \end{bmatrix} a_{vy} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_1 & \bar{R}_2 \end{bmatrix} (\lambda_r^{(+)}(R_1, R_2))^{-2} (\lambda_x^{(+)}(R_1, R_1))^{-2} \\
&\times (\lambda_u^{(-)}(R_1, \bar{R}_1))^{-1} \lambda_y^{(-)}(\bar{R}_1, R_2) \lambda_v^{(+)}(R_1, R_2)
\end{aligned} \tag{4.38}$$

$$\bullet \bar{P}_{([1],[1])}(\mathbf{7}_4^2; a, q) =$$

$$\frac{(1-a)}{a^3 q^2 (1-q)^2} \begin{pmatrix} 0 & -1 & 1 & -2 & 1 & -1 & 0 \\ 1 & -2 & 4 & -3 & 4 & -2 & 1 \\ 0 & -1 & 2 & -3 & 2 & -1 & 0 \end{pmatrix} \tag{4.39}$$

$$\bullet \bar{P}_{([1],[2])}(\mathbf{7}_4^2; a, q) =$$

$$\frac{(1-a)(1-aq)}{a^{7/2} q^{5/2} (1-q)^2 (1-q^2)} \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 2 & -1 & 2 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \end{pmatrix} \tag{4.40}$$

$$\bullet \bar{P}_{([1],[3])}(\mathbf{7}_4^2; a, q) =$$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^4 q^3 (1-q)^2 (1-q^2) (1-q^3)} \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & -1 & 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -2 & 3 & -1 & 3 & -2 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 & 2 & -2 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.41}$$

$$\bullet \bar{P}_{([1],[4])}(\mathbf{7}_4^2; a, q) =$$

$$\frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^{9/2} q^{7/2} (1-q)^2 (1-q^2) (1-q^3) (1-q^4)} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 & 3 & -1 & 3 & -1 & -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 & 1 & 1 & -2 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.42}$$

$$\bullet \bar{P}_{([2],[2])}(\mathbf{7}_4^2; a, q) =$$

$$\frac{(1-a)(1-aq)}{a^6 q^8 (1-q)^2 (1-q^2)^2} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 2 & -2 & 1 & 3 & -3 & 0 & 3 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -4 & 2 & 0 & -7 & 3 & 2 & -9 & 1 & 6 & -5 & -3 & 3 & 0 & -1 \\ 1 & -2 & 2 & 3 & -6 & 5 & 5 & -10 & 10 & 9 & -15 & 3 & 15 & -8 & -5 & 7 & 0 & -2 & 1 \\ -1 & 1 & 1 & -5 & 4 & 3 & -12 & 6 & 9 & -15 & -2 & 12 & -4 & -6 & 3 & 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 3 & -6 & 2 & 7 & -7 & -2 & 6 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.43}$$

- $\bar{P}_{([2],[3])}(\mathbf{7}_4^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^{13/2}q^{19/2}(1-q)^2(1-q^2)^2(1-q^3)} \times$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & -2 & 1 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & -1 & -2 & -2 & 0 & 2 & -3 & -5 & 1 & 4 & -1 & -4 & 0 & 2 & 0 & -1 \\ 1 & -1 & -1 & 3 & 0 & -1 & 2 & 0 & -2 & 1 & 9 & 3 & -9 & -2 & 11 & 3 & -6 & -3 & 4 & 3 & -2 & -1 & 1 \\ -1 & 0 & 2 & -1 & -2 & 2 & -1 & -4 & 2 & 6 & -3 & -10 & 1 & 9 & -1 & -7 & 0 & 3 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 2 & -1 & -1 & 2 & 1 & -1 & -3 & 2 & 3 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.44)$$

- $\bar{P}_{([2],[4])}(\mathbf{7}_4^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^7q^{11}(1-q)^2(1-q^2)^2(1-q^3)(1-q^4)} \times$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 1 & -1 & 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & -2 & -4 & -1 & 2 & -1 & 0 & -2 & -4 & 2 & 2 & -2 & 0 & -1 & -1 & 2 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 2 & 1 & 2 & -4 & -3 & 4 & 3 & 6 & 0 & -4 & 0 & 3 & 3 & 1 & -3 & -1 & 2 & 1 & 1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 & -1 & -1 & -3 & -1 & 6 & 0 & -2 & -2 & -4 & 1 & 3 & 0 & -2 & -2 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 2 & -1 & -2 & 2 & -1 & 0 & 2 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.45)$$

4.1.6 7_5^2 link

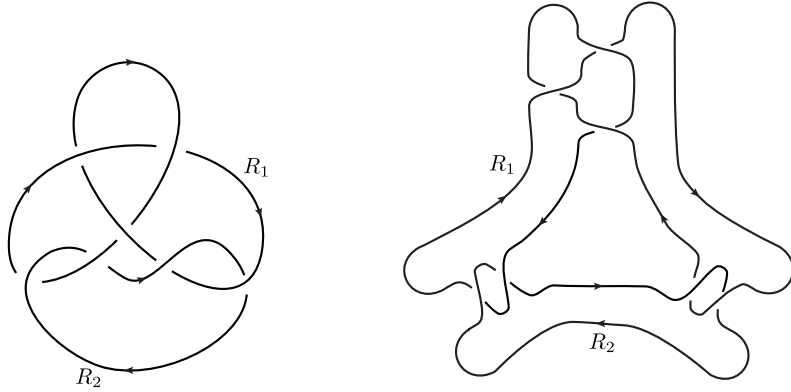


Figure 21. 7_5^2 link

$$\bar{P}_{R_1, R_2}(\mathbf{7}_5^2; a, q) = q^{3C_{R_1} + \frac{2\ell^1 \ell^2}{N}} \sum_{l, u, v, x, y, z} \frac{1}{\epsilon_l^{R_1, \bar{R}_1} \sqrt{\dim_q l}} \epsilon_y^{R_1, \bar{R}_1} \sqrt{\dim_q y} \epsilon_u^{\bar{R}_1, R_2}$$

$$\times \sqrt{\dim_q u} \epsilon_v^{\bar{R}_1, R_2} \sqrt{\dim_q v} a_{lx} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_1 & \bar{R}_1 \end{bmatrix} a_{zx} \begin{bmatrix} R_1 & R_1 \\ \bar{R}_1 & \bar{R}_1 \end{bmatrix} a_{zy} \begin{bmatrix} R_1 & R_1 \\ \bar{R}_1 & \bar{R}_1 \end{bmatrix}$$

$$\times a_{lu} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} a_{lv} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} \lambda_x^{(-)}(R_1, \bar{R}_1) \lambda_z^{(+)}(R_1, R_1) \lambda_y^{(-)}(R_1, \bar{R}_1)$$

$$\times (\lambda_u^{(-)}(\bar{R}_1, R_2))^2 (\lambda_v^{(-)}(\bar{R}_1, R_2))^2 \quad (4.46)$$

- $\overline{P}_{([1],[1])}(\mathbf{7}_5^2; a, q) =$

$$\frac{a(1-a)}{q(1-q)^2} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 3 & -3 & 0 \\ 2 & -4 & 6 & -4 & 2 \\ 1 & -3 & 4 & -3 & 1 \end{pmatrix} \quad (4.47)$$

- $\overline{P}_{([1],[2])}(\mathbf{7}_5^2; a, q) =$

$$\frac{a^{1/2}(1-a)(1-aq)}{q^{1/2}(1-q)^2(1-q^2)} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 & -3 & 0 \\ 1 & -1 & -1 & 4 & -2 & -1 & 2 \\ 1 & -2 & 1 & 1 & -2 & 1 & 0 \end{pmatrix} \quad (4.48)$$

- $\overline{P}_{([1],[3])}(\mathbf{7}_5^2; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{(1-q)^2(1-q^2)(1-q^3)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & -1 & 1 & -3 & 0 \\ 1 & -2 & 2 & -2 & 4 & -3 & 1 & -1 & 2 \\ 1 & -2 & 2 & -2 & 2 & -2 & 1 & 0 & 0 \end{pmatrix} \quad (4.49)$$

- $\overline{P}_{([1],[4])}(\mathbf{7}_5^2; a, q) =$

$$\frac{q^{1/2}(1-a)(1-aq)(1-aq^2)(1-aq^3)}{a^{1/2}(1-q)^2(1-q^2)(1-q^3)(1-q^4)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 & 0 & -1 & 1 & -3 & 0 \\ 1 & -2 & 1 & 1 & -2 & 4 & -3 & 0 & 1 & -1 & 2 \\ 1 & -2 & 2 & -1 & -1 & 2 & -2 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (4.50)$$

- $\overline{P}_{([2],[2])}(\mathbf{7}_5^2; a, q) =$

$$\frac{a^2(1-a)(1-aq)}{q^2(1-q)^2(1-q^2)^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & 3 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & -9 & 1 & 15 & -4 & -6 & 7 & 2 \\ 0 & 0 & 0 & -1 & -4 & 7 & 6 & -22 & 0 & 24 & -14 & -14 & 11 & 0 & -5 \\ 0 & 1 & -1 & -7 & 11 & 12 & -28 & -1 & 34 & -16 & -16 & 18 & 0 & -6 & 3 \\ 2 & -1 & -9 & 9 & 15 & -25 & -6 & 31 & -12 & -16 & 15 & 0 & -5 & 2 & 0 \\ 1 & -3 & 0 & 9 & -8 & -8 & 15 & -2 & -10 & 7 & 1 & -3 & 1 & 0 & 0 \end{pmatrix} \quad (4.51)$$

- $\overline{P}_{([2],[3])}(\mathbf{7}_5^2; a, q) =$

$$\frac{a^{3/2}(1-a)(1-aq)(1-aq^2)}{q^{3/2}(1-q)^2(1-q^2)^2(1-q^3)} \times$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 1 & 1 & -2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & -3 & -5 & 4 & 8 & 2 & -6 & -1 & 7 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 1 & 7 & -1 & -12 & -5 & 13 & 8 & -13 & -10 & 5 & 6 & -3 & -5 & 0 & 0 \\ 0 & 0 & 2 & -4 & -5 & 10 & 9 & -10 & -16 & 7 & 22 & -5 & -17 & 2 & 11 & 2 & -6 & -1 & 3 & 0 & 0 \\ 1 & 1 & -5 & -3 & 11 & 6 & -15 & -9 & 14 & 11 & -11 & -10 & 8 & 6 & -5 & -2 & 2 & 0 & 0 & 0 & 0 \\ 1 & -2 & -2 & 6 & 1 & -7 & -1 & 5 & 3 & -5 & -2 & 5 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.52)$$

$$\bullet \bar{P}_{([2],[4])}(\mathbf{7}_5^2; a, q) =$$

$$\frac{a(1-a)(1-aq)(1-aq^2)(1-aq^3)}{q(1-q)^2(1-q^2)^2(1-q^3)(1-q^4)} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & -1 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 & -1 & -1 & 3 & 6 & -2 & 0 & -1 & -1 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 3 & 0 & -1 & -7 & 0 & 5 & 1 & 3 & -9 & -5 & 4 & 0 & 3 & -3 & -5 \\ 0 & 0 & 1 & -1 & -3 & 0 & 6 & 3 & -6 & -2 & -3 & 2 & 11 & -4 & -4 & -2 & 0 & 8 & -1 & -2 & -1 & -1 & 3 \\ 1 & 0 & -3 & 0 & 2 & 3 & 2 & -8 & -1 & 4 & -1 & 6 & -4 & -5 & 4 & 0 & 2 & -2 & -2 & 2 & 0 & 0 & 0 \\ 1 & -2 & -1 & 4 & -1 & -1 & 0 & -2 & 2 & 0 & 1 & 0 & -3 & 3 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.53)$$

4.2 Three-component links

4.2.1 $T^{3,3}$ torus link

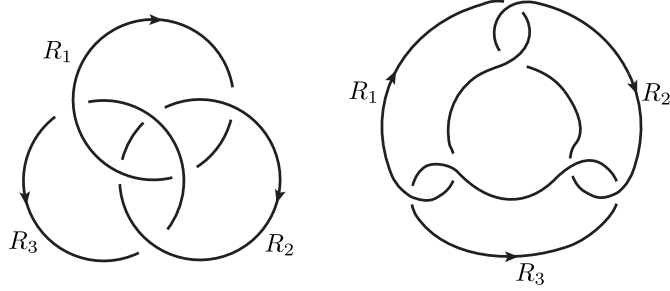


Figure 22. $T^{3,3}$ torus link

$$\begin{aligned} \bar{P}_{(R_1, R_2, R_3)}(T^{3,3}; a, q) &= q^{-\frac{\ell(1)\ell(2)}{N} - \frac{\ell(2)\ell(3)}{N} + \frac{\ell(1)\ell(3)}{N}} \sum_{l, x, y, z} \frac{1}{\epsilon_l^{R_1, \bar{R}_1} \sqrt{\dim_q l}} \epsilon_x^{\bar{R}_1, R_2} \sqrt{\dim_q x} \\ &\quad \times \epsilon_y^{R_2, R_3} \sqrt{\dim_q y} \epsilon_z^{R_1, R_3} \sqrt{\dim_q z} a_{lx} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} a_{yl} \begin{bmatrix} R_2 & R_3 \\ \bar{R}_3 & \bar{R}_2 \end{bmatrix} \\ &\quad \times a_{zl} \begin{bmatrix} R_1 & R_3 \\ \bar{R}_3 & \bar{R}_1 \end{bmatrix} (\lambda_x^{(-)}(\bar{R}_1, R_2))^{-2} (\lambda_y^{(+)}(R_2, R_3))^{-2} (\lambda_z^{(+)}(R_1, R_3))^2 \end{aligned} \quad (4.54)$$

The colored HOMFLY polynomials of the $(3,3)$ -torus link $T^{3,3}$ are symmetric under interchanging the representations R_1 and R_3 .

- $\bar{P}_{([1],[1],[1])}(T^{3,3}; a, q) =$

$$\frac{(1-a)}{a^{5/2}q^{1/2}(1-q)^3} \begin{pmatrix} 0 & 2 & -3 & 2 & 0 \\ -1 & 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix} \quad (4.55)$$

- $\bar{P}_{([1],[1],[2])}(T^{3,3}; a, q) = \bar{P}_{([2],[1],[1])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)}{a^3q(1-q)^3(1-q^2)} \begin{pmatrix} 0 & 0 & 2 & -2 & 0 & 1 & 0 \\ -1 & 1 & 0 & -2 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \end{pmatrix} \quad (4.56)$$

- $\bar{P}_{([1],[1],[3])}(T^{3,3}; a, q) = \bar{P}_{([3],[1],[1])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^{7/2}q^{3/2}(1-q)^3(1-q^2)(1-q^3)} \begin{pmatrix} 0 & 0 & 0 & 2 & -2 & 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 & -2 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (4.57)$$

- $\bar{P}_{([1],[2],[1])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)}{a^3q^2(1-q)^3(1-q^2)} \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 1 \\ -1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{pmatrix} \quad (4.58)$$

- $\bar{P}_{([1],[2],[2])}(T^{3,3}; a, q) = \bar{P}_{([2],[2],[1])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)}{a^{7/2}q^{7/2}(1-q)^3(1-q^2)^2} \begin{pmatrix} 0 & 0 & -2 & 2 & 2 & -5 & 1 & 3 & -2 & 0 \\ 1 & -1 & 1 & 3 & -3 & -1 & 4 & -1 & -1 & 1 \\ -1 & 0 & 1 & -2 & -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.59)$$

- $\bar{P}_{([1],[2],[3])}(T^{3,3}; a, q) = \bar{P}_{([3],[2],[1])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^4q^5(1-q)^3(1-q^2)^2(1-q^3)} \begin{pmatrix} 0 & 0 & 0 & -2 & 1 & 2 & 0 & -3 & -1 & 3 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 3 & -1 & -3 & 1 & 2 & 1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 0 & -2 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.60)$$

- $\bar{P}_{([1],[3],[1])}(T^{3,3}; a, q) =$

$$\frac{(1-aq)(1-a)(1-aq^2)}{a^{7/2}q^{7/2}(1-q)^3(1-q^2)(1-q^3)} \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & 2 & -2 & 1 \\ -1 & 0 & -1 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.61)$$

- $\overline{P}_{([1],[3],[2])}(T^{3,3}; a, q) = \overline{P}_{([2],[3],[1])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^4 q^4 (1-q)^3 (1-q^2)^2 (1-q^3)} \begin{pmatrix} 0 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 2 & -1 & -1 & 1 \\ -1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.62)$$

- $\overline{P}_{([1],[3],[3])}(T^{3,3}; a, q) = \overline{P}_{([3],[3],[1])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^{9/2} q^{17/2} (1-q)^3 (1-q^2)^2 (1-q^3)^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & -2 & -1 & 1 & 3 & 0 & -5 & 0 & 4 & 1 & -1 & -3 & 2 & 0 \\ 0 & -1 & 1 & -1 & -2 & -1 & 3 & 2 & -4 & -4 & 2 & 4 & 0 & -4 & 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 2 & 3 & -2 & -1 & -1 & 4 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -2 & -1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.63)$$

- $\overline{P}_{([2],[1],[2])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)}{a^{7/2} q^{3/2} (1-q)^3 (1-q^2)^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 1 & -3 & 1 & 5 & -3 & -1 & 2 \\ -1 & 1 & 1 & -3 & 0 & 2 & -2 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 & 0 & -1 & 1 & 0 & 0 \end{pmatrix} \quad (4.64)$$

- $\overline{P}_{([2],[1],[3])}(T^{3,3}; a, q) = \overline{P}_{([3],[1],[2])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^4 q^2 (1-q)^3 (1-q^2)^2 (1-q^3)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -3 & 0 & 3 & 2 & -1 & -2 & 1 & 1 \\ -1 & 1 & 1 & -1 & -2 & 0 & 2 & 0 & -2 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (4.65)$$

- $\overline{P}_{([2],[2],[2])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)}{a^5 q^4 (1-q)^3 (1-q^2)^3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 3 & -3 & -4 & 7 & 1 & -5 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 & -5 & 1 & 4 & -3 & -2 & 2 & 0 & -1 \\ 1 & -1 & 0 & 3 & 0 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & -2 & -2 & 2 & -1 & -2 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.66)$$

- $\overline{P}_{([2],[2],[3])}(T^{3,3}; a, q) = \overline{P}_{([3],[2],[2])}(T^{3,3}; a, q) =$

$$\frac{(1-a)(1-aq)(1-aq^2)}{a^{11/2} q^{11/2} (1-q)^3 (1-q^2)^3 (1-q^3)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -2 & -4 & 3 & 2 & 1 & -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & 2 & -4 & -5 & 2 & 6 & 0 & -6 & -2 & 4 & 1 & -2 \\ 1 & -1 & -1 & 2 & 2 & 0 & -3 & -1 & 6 & 3 & -3 & -3 & 2 & 3 & -1 & -1 & 1 \\ -1 & 0 & 2 & 0 & -3 & -1 & 2 & 1 & -2 & -2 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.67)$$

- $\overline{P}_{([2],[3],[2])}(T^{3,3}; a, q) = \frac{(1-a)(1-aq)(1-aq^2)}{a^{11/2}q^{15/2}(1-q)^3(1-q^2)^3(1-q^3)} \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & -1 & -4 & 4 & 3 & -3 & -3 & 1 & 5 & -2 & -3 & 2 & 0 \\ 0 & -1 & 1 & 0 & -4 & 0 & 4 & 1 & -5 & -3 & 4 & 2 & -2 & -2 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 & 2 & 2 & -1 & -2 & 2 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -2 & -1 & 1 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (4.68)

- $\overline{P}_{([2],[3],[3])}(T^{3,3}; a, q) = \overline{P}_{([3],[3],[2])}(T^{3,3}; a, q) = \frac{(1-a)(1-aq)(1-aq^2)}{a^6q^{10}(1-q)^3(1-q^2)^3(1-q^3)} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 2 & 4 & 0 & -7 & -4 & 9 & 5 & -5 & -6 & 1 & 5 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 3 & 4 & -3 & -7 & 3 & 8 & 2 & -7 & -4 & 5 & 3 & -1 & -2 & 0 & 1 \\ 0 & -1 & 1 & 0 & -2 & -2 & 1 & 2 & -2 & -6 & 0 & 3 & 2 & -4 & -3 & 2 & 1 & -1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 2 & 2 & 0 & -2 & 2 & 3 & 2 & -1 & -2 & 2 & 2 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -2 & -2 & 1 & 1 & -1 & -2 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (4.69)

- $\overline{P}_{([3],[1],[3])}(T^{3,3}; a, q) = \frac{(1-a)(1-aq)(1-aq^2)}{a^{9/2}q^{5/2}(1-q)^3(1-q^2)^2(1-q^3)^2} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 2 & -1 & -3 & -3 & 3 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -3 & 1 & 4 & 2 & -2 & -4 & 3 & 4 & 0 & -1 & -1 & 2 \\ -1 & 1 & 1 & 0 & -3 & -1 & 3 & 2 & -2 & -3 & 0 & 2 & -1 & -1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 2 & 1 & -2 & -1 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ (4.70)

- $\overline{P}_{([3],[2],[3])}(T^{3,3}; a, q) = \frac{(1-a)(1-aq)(1-aq^2)}{a^6q^7(1-q)^3(1-q^2)^3(1-q^3)^2} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 1 & 4 & 1 & -4 & -4 & 3 & 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -5 & -1 & 7 & 7 & -4 & -11 & 2 & 9 & 2 & -4 & -3 & 2 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & -4 & -4 & 1 & 6 & 0 & -8 & -4 & 3 & 4 & -2 & -3 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 1 & 3 & 0 & -3 & -2 & 3 & 4 & 2 & -2 & -1 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 1 & -3 & -3 & 2 & 4 & -1 & -4 & -1 & 2 & 0 & -2 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 2 & 1 & -2 & -1 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (4.71)

- $\overline{P}_{([3],[3],[3])}(T^{3,3}; a, q) = \frac{(1-a)(1-aq)(1-aq^2)}{a^{15/2}q^{23/2}(1-q)^3(1-q^2)^3(1-q^3)^3} \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 & -4 & -2 & 7 & 8 & -7 & -7 & -2 & 7 & 5 & -4 & -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 3 & 2 & -3 & -9 & -2 & 11 & 9 & -7 & -15 & -1 & 12 & 6 & -6 & -7 & 2 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 3 & 3 & 1 & -4 & -3 & 6 & 9 & 4 & -11 & -6 & 5 & 10 & 1 & -6 & -2 & 3 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -1 & -2 & 0 & 2 & 0 & -3 & -4 & -4 & 1 & 1 & 2 & -5 & -6 & -1 & 4 & 2 & -3 & -4 & 1 & 1 & 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 2 & 3 & -1 & -2 & 0 & 3 & 2 & 1 & 3 & 1 & -1 & -1 & 2 & 4 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 2 & -2 & -3 & 0 & 3 & 1 & -3 & -2 & 1 & 0 & -1 & -2 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 2 & 1 & -2 & -1 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (4.72)

4.2.2 6_1^3 link

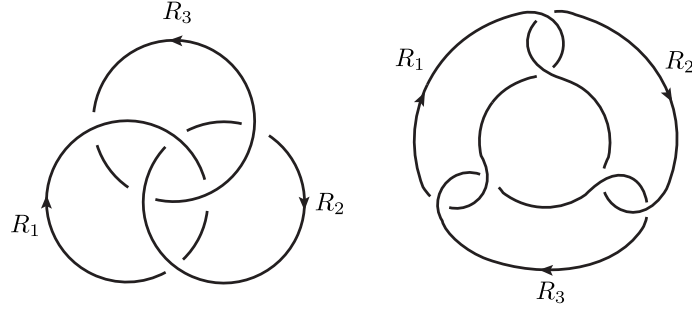


Figure 23. 6_1^3 link

$$\begin{aligned}
\bar{P}_{(R_1, R_2, R_3)}(\mathbf{6}_1^3; a, q) &= q^{-\frac{\ell(1)\ell(2)}{N} - \frac{\ell(2)\ell(3)}{N} - \frac{\ell(1)\ell(3)}{N}} \sum_{l, x, y, z} \frac{1}{\epsilon_l^{R_1, \bar{R}_1} \sqrt{\dim_q l}} \epsilon_x^{\bar{R}_1, R_2} \sqrt{\dim_q x} \\
&\times \epsilon_y^{\bar{R}_2, R_3} \sqrt{\dim_q y} \epsilon_z^{\bar{R}_1, R_3} \sqrt{\dim_q z} a_{lx} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} a_{ly} \begin{bmatrix} R_2 & \bar{R}_2 \\ R_3 & \bar{R}_3 \end{bmatrix} \\
&\times a_{lz} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_3 & \bar{R}_3 \end{bmatrix} (\lambda_x^{(-)}(\bar{R}_1, R_2))^2 (\lambda_y^{(-)}(\bar{R}_2, R_3))^2 (\lambda_z^{(-)}(\bar{R}_1, R_3))^2
\end{aligned} \tag{4.73}$$

The colored HOMFLY polynomials of the link $\mathbf{6}_1^3$ are symmetric under permutations over the representations (R_1, R_2, R_3) .

- $\bar{P}_{([1],[1],[1])}(\mathbf{6}_1^3; a, q) =$

$$\frac{a^{1/2}(1-a)}{q^{1/2}(1-q)^3} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 4 & -3 & 0 \\ 2 & -5 & 7 & -5 & 2 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} \tag{4.74}$$

- $\bar{P}_{([1],[1],[2])}(\mathbf{6}_1^3; a, q) =$

$$\frac{(1-a)(1-aq)}{(1-q)^3(1-q^2)} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 2 & -3 & 0 \\ 1 & -1 & -2 & 5 & -2 & -2 & 2 \\ 1 & -3 & 2 & 2 & -3 & 1 & 0 \end{pmatrix} \tag{4.75}$$

- $\bar{P}_{([1],[1],[3])}(\mathbf{6}_1^3; a, q) =$

$$\frac{q^{1/2}(1-a)(1-aq)(1-aq^2)}{a^{1/2}(1-q)^3(1-q^2)(1-q^3)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & -1 & 2 & -3 & 0 \\ 1 & -2 & 2 & -3 & 5 & -3 & 1 & -2 & 2 \\ 1 & -3 & 3 & -2 & 3 & -3 & 1 & 0 & 0 \end{pmatrix} \tag{4.76}$$

- $\overline{P}_{([1],[2],[2])}(\mathbf{6}_1^3; a, q) =$

$$\frac{(1-a)(1-aq)}{(1-q)^3(1-q^2)^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & -3 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 & 3 & 4 & -8 & -1 & 6 & -2 & -2 \\ 0 & 2 & -5 & 1 & 10 & -9 & -5 & 10 & -2 & -3 & 2 \\ 1 & -2 & -2 & 7 & -2 & -7 & 6 & 1 & -3 & 1 & 0 \end{pmatrix} \quad (4.77)$$

- $\overline{P}_{([1],[2],[3])}(\mathbf{6}_1^3; a, q) =$

$$\frac{q^{1/2}(1-a)(1-aq)(1-aq^2)}{a^{1/2}(1-q)^3(1-q^2)^2(1-q^3)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 3 & 0 & -5 & -2 & 4 & 2 & -2 & -2 \\ 0 & 1 & -1 & -3 & 3 & 5 & -3 & -6 & 1 & 6 & 0 & -3 & -1 & 2 \\ 1 & -2 & -1 & 3 & 2 & -3 & -3 & 2 & 3 & -1 & -2 & 1 & 0 & 0 \end{pmatrix} \quad (4.78)$$

- $\overline{P}_{([1],[3],[3])}(\mathbf{6}_1^3; a, q) =$

$$\frac{q^{1/2}(1-a)(1-aq)(1-aq^2)}{a^{1/2}(1-q)^3(1-q^2)^2(1-q^3)^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 3 & -1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -3 & -1 & -2 & 8 & 0 & 0 & -5 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & -3 & 3 & 4 & -1 & -7 & -6 & 11 & 4 & -2 & -9 & 0 & 5 & 0 & -1 & -2 \\ 0 & 0 & 2 & -5 & 1 & 3 & 7 & -7 & -10 & 4 & 11 & 2 & -9 & -5 & 7 & 1 & 0 & -3 & 2 & 0 \\ 1 & -2 & 0 & -1 & 6 & -1 & -5 & -3 & 4 & 6 & -3 & -5 & 2 & 1 & 2 & -3 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (4.79)$$

- $\overline{P}_{([2],[2],[2])}(\mathbf{6}_1^3; a, q) =$

$$\frac{a(1-a)(1-aq)}{q(1-q)^3(1-q^2)^3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 4 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & -11 & 3 & 16 & -8 & -8 & 7 & 2 \\ 0 & 0 & 0 & -1 & -3 & 9 & 3 & -25 & 8 & 28 & -20 & -14 & 15 & 1 & -5 \\ 0 & 1 & -2 & -5 & 15 & 4 & -35 & 12 & 38 & -28 & -17 & 23 & -1 & -7 & 3 \\ 2 & -3 & -9 & 18 & 11 & -41 & 7 & 43 & -27 & -18 & 22 & -1 & -6 & 2 & 0 \\ 1 & -4 & 2 & 12 & -17 & -8 & 28 & -8 & -17 & 12 & 2 & -4 & 1 & 0 & 0 \end{pmatrix} \quad (4.80)$$

- $\overline{P}_{([2],[2],[3])}(\mathbf{6}_1^3; a, q) =$

$$\frac{a^{1/2}(1-a)(1-aq)(1-aq^2)}{q^{1/2}(1-q)^3(1-q^2)^3(1-q^3)} \times$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 2 & 2 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & -5 & -6 & 5 & 9 & 2 & -10 & -3 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 & -3 & 2 & 8 & -3 & -14 & -4 & 19 & 11 & -17 & -13 & 7 & 10 & -2 & -5 \\ 0 & 0 & 2 & -5 & -3 & 13 & 6 & -16 & -18 & 15 & 29 & -11 & -25 & 2 & 15 & 4 & -8 & -2 & 3 \\ 1 & 0 & -6 & 0 & 16 & 1 & -25 & -6 & 27 & 13 & -22 & -15 & 14 & 10 & -7 & -3 & 2 & 0 & 0 \\ 1 & -3 & -1 & 10 & -3 & -13 & 3 & 12 & 3 & -13 & -3 & 10 & -1 & -3 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

homologies of ten-crossing thick knots in §3.2. The colored HOMFLY homology for links will be studied in [16].

Although the closed form expression [15] of the multiplicity-free $SU(N)$ quantum Racah coefficients extends to the scope in calculations of colored HOMFLY polynomials, we have not succeeded to obtain invariants for the knot 10_{161} and the link 7_6^2 . In addition, the information about colored HOMFLY polynomials beyond symmetric representations are very limited [29]. To deal with more complicated links and non-symmetric representations, the further study has to be undertaken for the $SU(N)$ quantum Racah coefficients with multiplicity structure. We hope to report on this issue in future.

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A Fusion matrices

In this appendix, we shall show the explicit expressions of the fusion matrices for some representations which are determined by quantum $6j$ -symbols in the companion paper [15].

$$1. R_1 = \square, \quad R_2 = \square\square$$

$$a_{ts} \begin{bmatrix} R_1 & \bar{R}_1 \\ R_2 & \bar{R}_2 \end{bmatrix} = \frac{1}{\sqrt{K_{23}}} \left(\begin{array}{c|ccc} & \kappa_0 & \kappa_1 & \kappa_2 \\ & s = \square & \square\square & \square\square\square \\ t = 0 & \sqrt{\dim_q \kappa_0} & -\sqrt{\dim_q \kappa_1} & \sqrt{\dim_q \kappa_2} \\ \square & -x_1 \sqrt{\frac{[N-1][N][N+1][N+2][N+3]}{[2][3]}} & x_2 \sqrt{\frac{[N][N+1][N+3]}{[3]}} & x_3 \sqrt{\frac{[N+2][N+3][N+4]}{[2]}} \\ \square\square & y_1 \sqrt{\frac{[N-1][N][N+2][N+3][N+4]}{[3]}} & y_2 \sqrt{\frac{[N][N+3][N+4]}{[2][3]}} & y_3 \sqrt{[N+1][N+2][N+3]} \end{array} \right) \quad (\text{A.1})$$

where $K_{23} = \dim_q R_1 \dim_q R_2 = \frac{[N]^2[N+1]^2[N+2]}{[3][2]^2}$ and the quantum dimensions are given by

$$\dim_q \kappa_0 = [N], \quad \dim_q \kappa_1 = \frac{[N-1][N][N+2]}{[2]}, \quad \dim_q \kappa_2 = \frac{[N-1][N]^2[N+1][N+4]}{[3][2]^2} \quad (\text{A.2})$$

and the variables x'_i 's and y'_i 's are given by

$$\begin{aligned} x_1 &= \frac{[2]}{[N+2]}, & x_2 &= \frac{([N-1][N+4]-[2])}{[2][N+3]}, & x_3 &= \frac{[N][N+1]}{[N+2][N+3]} \\ y_1 &= \frac{[N]}{[2][N+2]}, & y_2 &= \frac{[N][2]}{[N+3]}, & y_3 &= \frac{[N]}{[N+2][N+3]}. \end{aligned} \quad (\text{A.3})$$

$$a_{ts} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_1 & \bar{R}_2 \end{bmatrix} = \left(\begin{array}{c|ccc} t & s = \begin{array}{c} \kappa_0 \\ \square \end{array} & \begin{array}{c} \kappa_1 \\ \square \end{array} & \begin{array}{c} \kappa_2 \\ \square \end{array} \\ \hline \rho_{[3,2]} & z_1 \sqrt{\frac{[N-1][N][N+1][N+2]}{[4][3][2]}} & -\frac{[2]}{[N+1]} \sqrt{\frac{[N][N+1]}{[4][3]}} & \frac{[2]}{[N+2]} \sqrt{\frac{[N+2][N+4]}{[4][2]}} \\ \rho_{[4,1]} & -z_1 [2] \sqrt{\frac{[N-1][N+1][N+2][N+3]}{[2][3][5]}} & z_2 \sqrt{\frac{[N+1][N+3]}{[3][5]}} & z_3 \sqrt{\frac{[N][N+2][N+3][N+4]}{[2][5]}} \\ \rho_{[5]} & z_1 [3] \sqrt{\frac{[N+1][N+2][N+3][N+4]}{[5][4][3][2]}} & z_4 [3] \sqrt{\frac{[N-1][N+1][N+3][N+4]}{[5][4][3]}} & z_3 \sqrt{\frac{[N-1][N][N+2][N+3]}{[5][4][2]}} \end{array} \right), \quad (\text{A.4})$$

where $\rho_{[3,2]} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$, $\rho_{[4,1]} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$, $\rho_{[5]} = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}$ and the variables z_i 's are

$$\begin{aligned} z_1 &= \frac{[2]}{[N+1][N+2]}, & z_2 &= z_1 \left(\frac{[3][N+3][N+4] - [N][N-1]}{[N]+[3][N+4]} \right), \\ z_3 &= \frac{[2]}{[N+2][N+3]}, & z_4 &= \frac{[2]}{[N+1][N+3]}. \end{aligned} \quad (\text{A.5})$$

The quantum dimension of each representation for t are

$$\begin{aligned} \dim_q \rho_{[3,2]} &= \frac{[N-1][N]^2[N+1][N+2]}{[4][3][2]}, & \dim_q \rho_{[4,1]} &= \frac{[N-1][N][N+1][N+2][N+3]}{[5][3][2]}, \\ \dim_q \rho_{[5]} &= \frac{[N][N+1][N+2][N+3][N+4]}{[5]!}. \end{aligned} \quad (\text{A.6})$$

$$a_{ts} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_2 & \bar{R}_1 \end{bmatrix} = \frac{1}{\sqrt{K_{23}}} \left(\begin{array}{c|ccc} t & s = 0 & \begin{array}{c} \tilde{\rho}_1 \\ \square \end{array} & \begin{array}{c} \tilde{\rho}_2 \\ \square \end{array} \\ \hline \rho_{[3,2]} & \sqrt{\dim_q \rho_{[3,2]}} & -\frac{[N][N+1]}{[3]} \sqrt{\frac{[N+3]}{[4]}} & \frac{[N]}{[3]} \sqrt{\frac{[N+1][N+3][N+4]}{[2][4]}} \\ \rho_{[4,1]} & -\sqrt{\dim_q \rho_{[4,1]}} & -([N-2] + [N] - [N+4]) \frac{[N+1]}{[2][3]} \sqrt{\frac{[N]}{[5]}} & \frac{[2][N]}{[3]} \sqrt{\frac{[N][N+1][N+4]}{[2][5]}} \\ \rho_{[5]} & \sqrt{\dim_q \rho_{[5]}} & [N+1] \sqrt{\frac{[N-1][N][N+4]}{[4][5]}} & [N] \sqrt{\frac{[N-1][N][N+1]}{[2][4][5]}} \end{array} \right). \quad (\text{A.7})$$

2. $R = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$

$$a_{ts} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix} = \frac{1}{K_{33}} \left(\begin{array}{c|cccc} & s = \begin{array}{c} \tilde{\rho}_0 \\ \square \end{array} & \begin{array}{c} \tilde{\rho}_1 \\ \square \end{array} & \begin{array}{c} \tilde{\rho}_2 \\ \square \end{array} & \begin{array}{c} \tilde{\rho}_3 \\ \square \end{array} \\ \hline t = \tilde{\rho}_0 & -\sqrt{\dim_q \tilde{\rho}_0} & \sqrt{\dim_q \tilde{\rho}_1} & -\sqrt{\dim_q \tilde{\rho}_2} & \sqrt{\dim_q \tilde{\rho}_3} \\ \tilde{\rho}_1 & \sqrt{\dim_q \tilde{\rho}_1} & a & x & y \\ \tilde{\rho}_2 & -\sqrt{\dim_q \tilde{\rho}_2} & x & b & z \\ \tilde{\rho}_3 & \sqrt{\dim_q \tilde{\rho}_3} & y & z & c \end{array} \right) \quad (\text{A.8})$$

where $K_{33} = \dim_q R = \frac{[N][N+1][N+2]}{[2][3]}$ and the quantum dimensions of the representations are given by

$$\begin{aligned} \dim_q \tilde{\rho}_0 &= 1, & \dim_q \tilde{\rho}_1 &= [N-1][N+1], \\ \dim_q \tilde{\rho}_2 &= \frac{[N-1][N]^2[N+3]}{[2]^2}, & \dim_q \tilde{\rho}_3 &= \frac{[N-1][N]^2[N+1]^2[N+5]}{[3]^2[2]^2} \end{aligned} \quad (\text{A.9})$$

and we also have

$$\begin{aligned} a &= \frac{[N+1](1-[2][N-1][N+4])}{[3][N+3]}, & b &= \frac{[N](-1+[2][N][N+5])}{[3][N+4]}, & c &= \frac{[N][N+1]}{[N+3][N+4]}, \\ x &= \frac{[N](-[2]^2 + [N-1][N+5])}{[2][3][N+3]} \sqrt{[N+1][N+3]}, & y &= \frac{[N][N+1]}{[N+3][2]} \sqrt{[N+1][N+5]}, \\ z &= \frac{[N][N+1]}{[N+3][N+4]} \sqrt{[N+3][N+5]}. \end{aligned} \quad (\text{A.10})$$

$$a_{ts} \begin{bmatrix} R & R \\ \bar{R} & \bar{R} \end{bmatrix} = \frac{1}{K_{33}} \left(\begin{array}{c|cccc} & \hat{\rho}_0 & \hat{\rho}_1 & \hat{\rho}_2 & \hat{\rho}_3 \\ s = 0 & \square & \square & \square & \square \\ \hline t = & \begin{array}{c} \rho_{[6]} \\ \square \square \square \square \\ \rho_{[5,1]} \\ \square \square \square \\ \rho_{[4,2]} \\ \square \square \square \\ \rho_{[3,3]} \\ \square \square \square \end{array} & \begin{array}{c} -\sqrt{\dim_q \rho_{[6]}} \\ \sqrt{\dim_q \rho_{[5,1]}} \\ -\sqrt{\dim_q \rho_{[4,2]}} \\ \sqrt{\dim_q \rho_{[3,3]}} \end{array} & \begin{array}{c} \frac{[N][N+1]}{[2][3]} \sqrt{\frac{[N+1][N+2]}{[4]}} \\ x_1 \\ y_1 \\ z_1 \end{array} & \begin{array}{c} -\frac{[N][N+1]}{[2][3]} \sqrt{\frac{[N+2][N+3]}{[4]}} \\ x_2 \\ y_2 \\ z_2 \end{array} & \begin{array}{c} \frac{[N][N+1]}{[2][3]} \sqrt{\frac{[N+2][N+5]}{[4]}} \\ x_3 \\ y_3 \\ z_3 \end{array} \end{array} \right) \quad (\text{A.11})$$

The quantum dimensions of the **representations** are given by

$$\begin{aligned} \dim_q \rho_{[3,3]} &= \frac{[N-1][N]^2[N+1]^2[N+2]}{[2]^2[3]^2[4]}, & \dim_q \rho_{[4,2]} &= \frac{[N-1][N]^2[N+1][N+2][N+3]}{[2]^2[4][5]}, \\ \dim_q \rho_{[5,1]} &= \frac{[N-1][N][N+1][N+2][N+3][N+4]}{[2][3][4][6]}, & \dim_q \rho_{[6]} &= \frac{[N][N+1][N+2][N+3][N+4][N+5]}{[2][3][4][5][6]}. \end{aligned} \quad (\text{A.12})$$

We also have

$$\begin{aligned} x_1 &= \frac{[N][N+1]([N-1]-[2][N+4])}{[2][3][N+3]} \sqrt{\frac{[N+2][N+3]}{[4][5]}}, & x_2 &= \frac{[N](-[2][N]+[N+5])}{[2][3]} \sqrt{\frac{[N+1][N+2]}{[4][5]}}, \\ x_3 &= \frac{[N][N+1]}{[2][3]} \sqrt{\frac{[N+1][N+2][N+3][N+5]}{[4][5]}}, & y_1 &= \frac{[N+1]([N+3]-[2][N])}{[N+3]} \sqrt{\frac{[N][N+2][N+3][N+4]}{[2][3][4][6]}}, \\ y_2 &= \frac{[N]([N+5][N+2]+[2N+4])}{[N+4][N+2]} \sqrt{\frac{[N][N+1][N+2][N+4]}{[2][3][4][6]}}, & y_3 &= \frac{[N][N+1]}{[N+4][N+3]} \sqrt{[N][N+1][N+2][N+3][N+4]}, \\ z_1 &= \frac{[3][N+1]}{[N+3]} \sqrt{\frac{[N-1][N][N+2][N+3][N+4][N+5]}{[2][3][4][5][6]}}, & z_2 &= \frac{[3][N]}{[N+4]} \sqrt{\frac{[N-1][N][N+1][N+2][N+4][N+5]}{[2][3][4][5][6]}}, \\ z_3 &= \frac{[N][N+1]}{[N+3][N+4]} \sqrt{\frac{[N-1][N][N+1][N+2][N+3][N+4]}{[4]}}. \end{aligned} \quad (\text{A.13})$$

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